

Lecture 15

Doubly Robust Estimator

Outline

- Doubly robust estimator
 - Interpretation
 - Comparison with IPW and out regression
 - A well-known simulation study
- Suggested Reading: Peng's book Chapter 12

Two ways to estimate the ATE

- Outcome regression

$$\begin{aligned}\tau(\mathbf{x}) &= \mathbb{E}(Y_i(1) \mid \mathbf{X}_i = \mathbf{x}, W_i = 1) - \mathbb{E}(Y_i(0) \mid \mathbf{X}_i = \mathbf{x}, W_i = 0) \\ &= \mathbb{E}(Y_i^{\text{obs}} \mid \mathbf{X}_i = \mathbf{x}, W_i = 1) - \mathbb{E}(Y_i^{\text{obs}} \mid \mathbf{X}_i = \mathbf{x}, W_i = 0) \\ &= \mu_1(\mathbf{x}) - \mu_0(\mathbf{x})\end{aligned}$$

- relies on a correctly specified model for the outcomes depending on \mathbf{X}_i

$$\hat{\tau}^{\text{reg}} = \frac{1}{N} \sum_{i=1}^N \hat{\mu}_1(\mathbf{X}_i) - \frac{1}{N} \sum_{i=1}^N \hat{\mu}_0(\mathbf{X}_i)$$

- IPW / Matching

$$\tau = \mathbb{E}\left(\frac{Y_i^{\text{obs}} W_i}{e(\mathbf{X}_i)}\right) - \mathbb{E}\left(\frac{Y_i^{\text{obs}} \cdot (1 - W_i)}{1 - e(\mathbf{X}_i)}\right)$$

- relies on a correctly specified model for the propensity score

$$\hat{\tau}^{\text{IPW}} = \frac{1}{N} \sum_{i=1}^N \left\{ \frac{W_i Y_i^{\text{obs}}}{\hat{e}(\mathbf{X}_i)} - \frac{(1 - W_i) Y_i^{\text{obs}}}{1 - \hat{e}(\mathbf{X}_i)} \right\}$$

Doubly robust estimator

- Can we provide a good estimate if either model is correct?
- Doubly robust estimator: provide a good estimate of the propensity score when either the outcome or the propensity score model is correct

- Define

$$f(1, \mathbf{X}_i, Y_i^{\text{obs}}) = \frac{Y_i^{\text{obs}} 1_{W_i=1}}{\tilde{e}(\mathbf{X}_i)} - \frac{1_{W_i=1} - \tilde{e}(\mathbf{X}_i)}{\tilde{e}(\mathbf{X}_i)} \tilde{\mu}_1(\mathbf{X}_i)$$

$$f(0, \mathbf{X}_i, Y_i^{\text{obs}}) = \frac{Y_i^{\text{obs}} 1_{W_i=0}}{1 - \tilde{e}(\mathbf{X}_i)} - \frac{1_{W_i=0} - (1 - \tilde{e}(\mathbf{X}_i))}{1 - \tilde{e}(\mathbf{X}_i)} \tilde{\mu}_0(\mathbf{X}_i)$$

- $\tilde{e}(\mathbf{X}_i), \tilde{\mu}_w(\mathbf{X}_i)$: our working models (model under our model assumption)
- $e(\mathbf{X}_i), \mu_w(\mathbf{X}_i)$: true model that we don't know
 - If we correctly specify the **propensity score model**, then $\tilde{e}(\mathbf{X}_i) = e(\mathbf{X}_i)$
 - If we correctly specify the **outcome model**, then $\tilde{\mu}_w(\mathbf{X}_i) = \mu_w(\mathbf{X}_i)$

Interpretation

$$f(1, \mathbf{X}_i, Y_i^{\text{obs}}) = \underbrace{\frac{Y_i^{\text{obs}} \mathbf{1}_{W_i=1}}{\tilde{e}(\mathbf{X}_i)}}_{\text{IPW estimate of } \mathbb{E}(Y_i(1) | \mathbf{X}_i)} - \underbrace{\frac{\mathbf{1}_{W_i=1} - \tilde{e}(\mathbf{X}_i)}{\tilde{e}(\mathbf{X}_i)} \tilde{\mu}_1(\mathbf{X}_i)}_{\text{Adjust for bias if the propensity score model is incorrect (if PS model is correct, then this part has expectation 0)}}$$

IPW estimate of $\mathbb{E}(Y_i(1) | \mathbf{X}_i)$

Adjust for bias if the propensity score model is incorrect
(if PS model is correct, then this part has expectation 0)

An equivalent expression:

$$f(1, \mathbf{X}_i, Y_i^{\text{obs}}) = \underbrace{\tilde{\mu}_1(\mathbf{X}_i)}_{\text{Outcome regression estimate of } \mathbb{E}(Y_i(1) | \mathbf{X}_i)} + \underbrace{\frac{\mathbf{1}_{W_i=1}}{\tilde{e}(\mathbf{X}_i)} (Y_i^{\text{obs}} - \tilde{\mu}_1(\mathbf{X}_i))}_{\text{Adjust for bias if the outcome regression model is incorrect (if PS model is correct, then this part has expectation 0)}}$$

Outcome regression estimate of $\mathbb{E}(Y_i(1) | \mathbf{X}_i)$

Adjust for bias if the outcome regression model is incorrect
(if PS model is correct, then this part has expectation 0)

Doubly robust property

- Double robust property

$$\mathbb{E} [f(1, \mathbf{X}_i, Y_i^{\text{obs}}) | \mathbf{X}_i] = \frac{(\mu_1(\mathbf{X}_i) - \tilde{\mu}_1(\mathbf{X}_i)) (e(\mathbf{X}_i) - \tilde{e}(\mathbf{X}_i))}{\tilde{e}(\mathbf{X}_i)} + \mu_1(\mathbf{X}_i)$$

$$\mathbb{E} [f(0, \mathbf{X}_i, Y_i^{\text{obs}}) | \mathbf{X}_i] = \frac{(\mu_0(\mathbf{X}_i) - \tilde{\mu}_0(\mathbf{X}_i)) (\tilde{e}(\mathbf{X}_i) - e(\mathbf{X}_i))}{1 - \tilde{e}(\mathbf{X}_i)} + \mu_0(\mathbf{X}_i)$$

- If either the outcome or propensity score model is correct, we have

$$\mathbb{E} \left(f(w, \mathbf{X}_i, Y_i^{\text{obs}}) \right) = \mathbb{E}(Y_i(w) | \mathbf{X}_i)$$

- The DR estimator

$$\hat{\tau}^{\text{dr}} = \frac{1}{N} \sum_{i=1}^N \left\{ \frac{W_i Y_i^{\text{obs}}}{\hat{e}(\mathbf{X}_i)} - \frac{W_i - \hat{e}(\mathbf{X}_i)}{\hat{e}(\mathbf{X}_i)} \hat{\mu}_1(\mathbf{X}_i) \right\} - \frac{1}{N} \sum_{i=1}^N \left\{ \frac{(1 - W_i) Y_i^{\text{obs}}}{1 - \hat{e}(\mathbf{X}_i)} - \frac{\hat{e}(\mathbf{X}_i) - W_i}{1 - \hat{e}(\mathbf{X}_i)} \hat{\mu}_0(\mathbf{X}_i) \right\}$$

Comparison with other estimators

- The DR estimator

$$\hat{\tau}^{\text{dr}} = \frac{1}{N} \sum_{i=1}^N \left\{ \frac{W_i Y_i^{\text{obs}}}{\hat{e}(\mathbf{X}_i)} - \frac{W_i - \hat{e}(\mathbf{X}_i)}{\hat{e}(\mathbf{X}_i)} \hat{\mu}_1(\mathbf{X}_i) \right\} - \frac{1}{N} \sum_{i=1}^N \left\{ \frac{(1 - W_i) Y_i^{\text{obs}}}{1 - \hat{e}(\mathbf{X}_i)} - \frac{\hat{e}(\mathbf{X}_i) - W_i}{1 - \hat{e}(\mathbf{X}_i)} \hat{\mu}_0(\mathbf{X}_i) \right\}$$

- Comparison with the IPW estimator

$$\hat{\tau}^{\text{dr}} = \hat{\tau}^{\text{IPW}} - \frac{1}{N} \sum_{i=1}^N \frac{W_i - \hat{e}(\mathbf{X}_i)}{\hat{e}(\mathbf{X}_i)} \hat{\mu}_1(\mathbf{X}_i) + \frac{1}{N} \sum_{i=1}^N \frac{\hat{e}(\mathbf{X}_i) - W_i}{1 - \hat{e}(\mathbf{X}_i)} \hat{\mu}_0(\mathbf{X}_i)$$

- Comparison with the outcome regression estimator

$$\hat{\tau}^{\text{dr}} = \hat{\tau}^{\text{reg}} + \frac{1}{N} \sum_{i=1}^N \frac{W_i}{\hat{e}(\mathbf{X}_i)} \left(Y_i^{\text{obs}} - \hat{\mu}_1(\mathbf{X}_i) \right) - \frac{1}{N} \sum_{i=1}^N \frac{1 - W_i}{1 - \hat{e}(\mathbf{X}_i)} \left(Y_i^{\text{obs}} - \hat{\mu}_0(\mathbf{X}_i) \right)$$

- Use bootstrap to compute the variance of $\hat{\tau}^{\text{dr}}$

A simulation study (Kang and Schafer. 2007. Statistical Science)

- Setup:
 - 4 covariates Z_i : all are i.i.d. standard normal
 - Outcome model: linear model

$$y_i = 210 + 27.4z_{i1} + 13.7z_{i2} + 13.7z_{i3} + 13.7z_{i4} + \varepsilon_i$$

- Propensity score model: logistic model with linear predictors

$$\pi_i = \text{expit}(-z_{i1} + 0.5z_{i2} - 0.25z_{i3} - 0.1z_{i4})$$

- Misspecification induced by measurement error:

$$x_{i1} = \exp(z_{i1}/2),$$

$$x_{i2} = z_{i2}/(1 + \exp(z_{i1})) + 10,$$

$$x_{i3} = (z_{i1}z_{i3}/25 + 0.6)^3,$$

$$x_{i4} = (z_2 + z_4 + 20)^2.$$

- Corresponding outcome regression / propensity score model is mis-specified if
Observe X_i instead of Z_i

A simulation study (Kang and Schafer. 2007. Statistical Science)

- The simulation reveals the deteriorating performance of propensity score weighting methods when the model is mis-specified
- Weighting estimators to be evaluated:
 - HT: IPW in the original form
 - IPW: IPW with normalized weights
 - WLS: Weighted least squares regression with covariates
 - IPW with normalization weights with some heuristic adjustment of covariates to improve efficiency
 - Not doubly robust
 - DR: Doubly-robust estimator

Results: if the propensity score model is correct

Sample size	Estimator	Bias		RMSE	
		logit	True	logit	True
(1) Both models correct					
$n = 200$	HT	0.33	1.19	12.61	23.93
	IPW	-0.13	-0.13	3.98	5.03
	WLS	-0.04	-0.04	2.58	2.58
	DR	-0.04	-0.04	2.58	2.58
$n = 1000$	HT	0.01	-0.18	4.92	10.47
	IPW	0.01	-0.05	1.75	2.22
	WLS	0.01	0.01	1.14	1.14
	DR	0.01	0.01	1.14	1.14
(2) Propensity score model correct					
$n = 200$	HT	-0.05	-0.14	14.39	24.28
	IPW	-0.13	-0.18	4.08	4.97
	WLS	0.04	0.04	2.51	2.51
	DR	0.04	0.04	2.51	2.51
$n = 1000$	HT	-0.02	0.29	4.85	10.62
	IPW	0.02	-0.03	1.75	2.27
	WLS	0.04	0.04	1.14	1.14
	DR	0.04	0.04	1.14	1.14

- Normalizing weights can help a lot in reducing the variance
- WLS indeed gain efficiency even if outcome model is not linear
- Use the true propensity score is worse than using the estimated propensity score when the propensity score model is correct

Results: if the propensity score model is incorrect

Sample size	Estimator	Bias		RMSE	
		logit	True	logit	True
(3) Outcome model correct					
$n = 200$	HT	24.25	-0.18	194.58	23.24
	IPW	1.70	-0.26	9.75	4.93
	WLS	-2.29	0.41	4.03	3.31
	DR	-0.08	-0.10	2.67	2.58
$n = 1000$	HT	41.14	-0.23	238.14	10.42
	IPW	4.93	-0.02	11.44	2.21
	WLS	-2.94	0.20	3.29	1.47
	DR	0.02	0.01	1.89	1.13
(4) Both models incorrect					
$n = 200$	HT	30.32	-0.38	266.30	23.86
	IPW	1.93	-0.09	10.50	5.08
	WLS	-2.13	0.55	3.87	3.29
	DR	-7.46	0.37	50.30	3.74
$n = 1000$	HT	101.47	0.01	2371.18	10.53
	IPW	5.16	0.02	12.71	2.25
	WLS	-2.95	0.37	3.30	1.47
	DR	-48.66	0.08	1370.91	1.81

When only the outcome model is wrong

- Double robust estimator perform better when outcome model is correct but propensity score model is wrong
- WLS improves over IPW but not as good as DR

When both models are wrong

- Double robust estimator can perform worse when both models are wrong (maybe we should also normalize the weights in DR)