# Lecture 17
# Sensitivity Analysis

# Outline

- Sensitivity analysis
  - Bound under no assumptions
  - Bound for the smoking example
  - A model-based approach
  - Rosenbaum sensitivity analysis

# Sensitivity analysis

- Most often, validity of unconfoundedness can not be easily checked. Alternatively, one should check sensitivity of a causal analysis to unconfoundedness

- **Sensitivity analysis** aims at assessing the bias of causal effect estimates when the unconfoundedness assumption is assumed to fail in some specific and meaningful ways

- Sensitivity is different from testing – unconfoundedness is intrinsically non-testable, more of a "insurance" check

- Sensitivity analysis in causal inference dates back to the Hill-Fisher debate on causation between smoking and lung cancer, and first formalized in Cornfield (1959, JNCI)

# Bounds under no assumptions

- Consider a simple case where: 1. no covariates; 2. binary outcome
- We are interested in the ATE

$$\tau_{\mathrm{sp}} = \mu_{\mathrm{t}} - \mu_{\mathrm{c}},$$

where

$$\mu_{\mathrm{t}} = \mathbb{E}\left[Y_i(1)\right] = p \cdot \mu_{\mathrm{t},1} + (1-p) \cdot \mu_{\mathrm{t},0},$$

and

$$\mu_{\mathrm{c}} = \mathbb{E}\left[Y_i(0)\right] = p \cdot \mu_{\mathrm{c},1} + (1-p) \cdot \mu_{\mathrm{c},0}.$$

$$\mu_{t,1} = \mathbb{E}[Y_i(1)|W_i = 1]$$

$$\mu_{t,0} = \mathbb{E}[Y_i(1)|W_i = 0]$$

$$\mu_{c,1} = \mathbb{E}[Y_i(0)|W_i = 1]$$

$$\mu_{c,0} = \mathbb{E}[Y_i(0)|W_i = 0]$$

$$p = P(W_i = 1)$$

Identifiable from observed data

Bound the unknown $\mu_{t,0}$ and $\mu_{c,1}$ by $[0, 1]$ as the outcome is binary

# Bounds under no assumptions

- So we get the bounds
$$\mu_t \in \left[p \cdot \mu_{t,1}, p \cdot \mu_{t,1} + (1-p)\right]$$
$$\mu_c \in \left[(1-p) \cdot \mu_{c,0}, (1-p) \cdot \mu_{c,0} + p\right]$$
- The the bound of ATE $\tau = \tau_{sp} = \mu_t - \mu_c$ is
$$\tau \in \left[p \cdot \mu_{t,1} - (1-p) \cdot \mu_{c,0} - p, p \cdot \mu_{t,1} + (1-p) - (1-p) \cdot \mu_{c,0}\right]$$
- Unfortunately, because we don't have any assumptions at all, this bound is not very informative
  - $\tau^{upper} - \tau^{lower}$
  $$= p\mu_{t,1} + (1-p) - (1-p)\mu_{c,0} - p\mu_{t,1} + (1-p)\mu_{c,0} + p \equiv 1$$
  - By definition, $\tau^{upper} \leq 1$ and $\tau^{lower} \geq -1$ , bound always covers 0
  - Better than the naive bound $[-1,1]$

# The Imbens-Rubin-Sacerdote lottery data

[Estimating the effect of unearned income on labor earnings, savings, and consumption: Evidence from a survey of lottery players. *American economic review*, 2001]

- Goal: Estimate magnitude of lottery prizes (unearned income) on economic behavior, including labor supply, consumption and savings

- Data collection:
  - “Winners”: individuals who had played and won large sums of money in the Massachsetts lottery
  - “Losers”: individuals who played the lottery and had won only small prizes

- We analyze a subset of $N_t = 259$ and $N_c = 237$ individuals with complete answers

# Result on the lottery data

- Binary outcome: whether the earning after treatment is positive or not

- Estimated quantities: $\hat{p} = \frac{N_t}{N} = 0.4675$, $\hat{\mu}_{t,1} = \bar{Y}_t^{\text{obs}} = 0.4106$ and $\hat{\mu}_{c,0} = \bar{Y}_c^{\text{obs}} = 0.5349$

- Plug in these quantities into our bound:

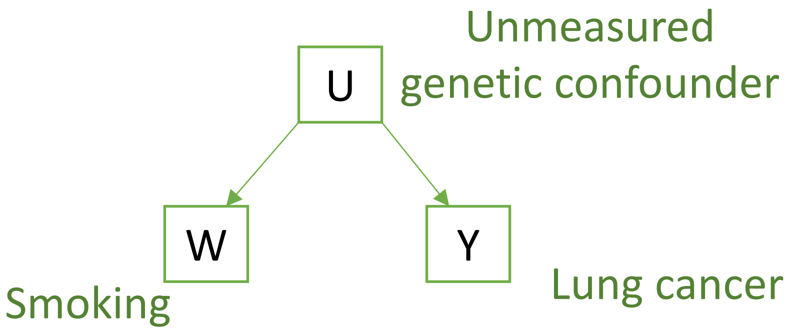
$$\tau \in [-0.56, 0.44]$$

- The two-sample difference estimate: $\bar{Y}_t^{\text{obs}} - \bar{Y}_c^{\text{obs}} = -0.124$

# Sensitivity analysis bound: a more useful example

The smoking on lung cancer effect example (Cornfield et al. 1959 JNCI)

- Fisher argued the association between smoking and lung cancer may be due to a common gene that causes both




- Observed association between smoking and lung cancer
  - Risk ratio

$$RR_{WY} = \frac{P\left[Y_i^{\text{obs}} = 1 \middle| W_i = 1\right]}{P\left[Y_i^{\text{obs}} = 1 \middle| W_i = 0\right]}$$

  - Observed risk ratio $RR_{WY} \approx 9$
  - Can this be fully explained by $U$?

# Sensitivity analysis bound: a more useful example

- Assume that $U_i$ are binary variables
- Define

$$p_0 = P[U_i = 1|W_i = 0], \qquad p_1 = P[U_i = 1|W_i = 1]$$

- $RR_{WU} = \frac{p_1}{p_0}$

- If there is no causal effect of smoking on lung cancer, then $Y_i(0) = Y_i(1) = Y_i$
  $P\left[Y_i^{\text{obs}} = 1 \middle| W_i = 0, U_i = 0\right] = P\left[Y_i^{\text{obs}} = 1 \middle| W_i = 1, U_i = 0\right] = P[Y_i = 1|U_i = 0] = r_0,$
  $P\left[Y_i^{\text{obs}} = 1 \middle| W_i = 0, U_i = 1\right] = P\left[Y_i^{\text{obs}} = 1 \middle| W_i = 1, U_i = 1\right] = P[Y_i = 1|U_i = 1] = r_1$

- Then we have

$$RR_{WY} = \frac{P\left[Y_i^{\text{obs}} = 1 \middle| W_i = 0, U_i = 0\right]}{P\left[Y_i^{\text{obs}} = 1 \middle| W_i = 0, U_i = 1\right]} = \frac{r_0(1 - p_1) + r_1 p_1}{r_0(1 - p_0) + r_1 p_0}$$

# Sensitivity analysis bound: a more useful example

$$RR_{WY} = \frac{r_0(1-p_1)+r_1p_1}{r_0(1-p_0)+r_1p_0}, \qquad RR_{WU} = \frac{P[U_i=1|W_i=1]}{P[U_i=1|W_i=0]} = \frac{p_1}{p_0}$$

- As $p_1 \geq p_0$ because we observe $RR_{WY} > 1$, then (from some math)

$$RR_{WY} = \frac{r_0(1-p_1)+r_1p_1}{r_0(1-p_0)+r_1p_0} \leq \frac{p_1}{p_0} = RR_{WU}$$

- Cornfield showed that if Fisher is right, we have $RR_{WU} \geq RR_{WY} \approx 9$

- Such a genetic confounder might be too strong to be realistic

- If we believe that such genetic confounder does not exist, then smoking should have a causal effect on lung cancer

# Another sensitivity analysis idea: base on a model

Idea:

observed ↓ ↙ unobserved

$$W_i \perp \big(Y_i(0), Y_i(1)\big) \mid \boldsymbol{X}_i, U_i$$

- How sensitive is our estimate of causal effect to the presence of $U_i$?

- A model-based approach (Rosenbaum and Rubin, 1983 JRSS-B)
  - Consider the scenario that $Y_i(w)$ is binary
  - Assume that the unmeasured confounding is binary
  - Build the following model

$$U_i \sim \text{Bernoulli}(q)$$
$$\text{logit}(P[W_i = 1 \mid \boldsymbol{X}_i, U_i]) = \gamma_0 + \boldsymbol{X}_i^T \boldsymbol{\kappa} + \gamma_1 U_i \quad \text{Propensity score model}$$
$$\text{logit}(P[Y_i(0) = 1 \mid \boldsymbol{X}_i, U_i]) = \beta_0 + \boldsymbol{X}_i^T \boldsymbol{b_0} + \beta_0 U_i$$
$$\text{logit}(P[Y_i(1) = 1 \mid \boldsymbol{X}_i, U_i]) = \alpha_0 + \boldsymbol{X}_i^T \boldsymbol{b_1} + \alpha_1 U_i$$

Outcome regression model

# Another sensitivity analysis idea: base on a model

$$U_i \sim \text{Bernoulli}(q)$$
$$\text{logit}(P[W_i = 1 |\ \boldsymbol{X}_i, U_i]) = \gamma_0 + \ \boldsymbol{X}_i^T \boldsymbol{\kappa} + \gamma_1 U_i$$
$$\text{logit}(P[Y_i(0) = 1 |\ \boldsymbol{X}_i, U_i]) = \beta_0 + \ \boldsymbol{X}_i^T \boldsymbol{b_0} + \beta_1 U_i$$
$$\text{logit}(P[Y_i(1) = 1 |\ \boldsymbol{X}_i, U_i]) = \alpha_0 + \ \boldsymbol{X}_i^T \boldsymbol{b_1} + \alpha_1 U_i$$

Propensity score model

Outcome regression model

- Sensitivity parameters: $(q, \gamma_1, \beta_1, \alpha_1)$
- Sensitivity parameters can not be estimated as unmeasured confounder $U_i$ is unobserved
- Sensitivity analysis: Set the sensitivity parameters to different values and see how estimates of causal effects change

# An example of calculation

$$U_i \sim \text{Bernoulli}(q)$$
$$\text{logit}(P[W_i = 1 | \boldsymbol{X}_i, U_i]) = \gamma_0 + \gamma_1 U_i$$
$$\text{logit}(P[Y_i(0) = 1 | \boldsymbol{X}_i, U_i]) = \beta_0 + \beta_1 U_i$$
$$\text{logit}(P[Y_i(1) = 1 | \boldsymbol{X}_i, U_i]) = \alpha_0 + \alpha_1 U_i$$

- Consider the simpler case where there is no $\boldsymbol{X}_i$
- Our observed data provides estimates of $\mathrm{p} = \mathbb{E}(W_i) = \mathbb{P}(W_i = 1)$, $\mu_{t,1} = \mathbb{E}\left[Y_i^{obs} \middle| W_i = 1\right]$ and $\mu_{c,0} = \mathbb{E}\left[Y_i^{obs} \middle| W_i = 0\right]$

$$p = q \cdot \frac{\exp(\gamma_0 + \gamma_1)}{1 + \exp(\gamma_0 + \gamma_1)} + (1 - q) \cdot \frac{\exp(\gamma_0)}{1 + \exp(\gamma_0)}$$

$$\begin{aligned}
\mu_{\mathrm{t},1} &= \Pr(U_i = 1 | W_i = 1) \cdot \mathbb{E}[Y_i(1) | W_i = 1, U_i = 1] \\
&\quad + (1 - \Pr(U_i = 1 | W_i = 1)) \cdot \mathbb{E}[Y_i(1) | W_i = 1, U_i = 0] \\
&= \frac{q \cdot \frac{\exp(\gamma_0 + \gamma_1)}{1 + \exp(\gamma_0 + \gamma_1)}}{q \cdot \frac{\exp(\gamma_0 + \gamma_1)}{1 + \exp(\gamma_0 + \gamma_1)} + (1 - q) \cdot \frac{\exp(\gamma_0)}{1 + \exp(\gamma_0)}} \cdot \frac{\exp(\alpha_0 + \alpha_1)}{1 + \exp(\alpha_0 + \alpha_1)} \\
&\quad + \frac{(1 - q) \cdot \frac{\exp(\gamma_0)}{1 + \exp(\gamma_0)}}{q \cdot \frac{\exp(\gamma_0 + \gamma_1)}{1 + \exp(\gamma_0 + \gamma_1)} + (1 - q) \cdot \frac{\exp(\gamma_0)}{1 + \exp(\gamma_0)}} \cdot \frac{\exp(\alpha_0)}{1 + \exp(\alpha_0)},
\end{aligned}$$

$$\begin{aligned}
\mu_{\mathrm{c},0} &= \frac{q \cdot \frac{1}{1 + \exp(\gamma_0 + \gamma_1)}}{q \cdot \frac{1}{1 + \exp(\gamma_0 + \gamma_1)} + (1 - q) \cdot \frac{1}{1 + \exp(\gamma_0)}} \cdot \frac{\exp(\beta_0 + \beta_1)}{1 + \exp(\beta_0 + \beta_1)} \\
&\quad + \frac{(1 - q) \cdot \frac{1}{1 + \exp(\gamma_0)}}{q \cdot \frac{1}{1 + \exp(\gamma_0 + \gamma_1)} + (1 - q) \cdot \frac{1}{1 + \exp(\gamma_0)}} \cdot \frac{\exp(\beta_0)}{1 + \exp(\beta_0)}.
\end{aligned}$$

Given any value of $(q, \gamma_1, \beta_1, \alpha_1)$, we can solve the three equations to estimate $(\gamma_0, \beta_0, \alpha_0)$

# An example of calculation

$$U_i \sim \text{Bernoulli}(q)$$
$$\text{logit}(P[W_i = 1|U_i]) = \gamma_0 + \gamma_1 U_i$$
$$\text{logit}(P[Y_i(0) = 1|U_i]) = \beta_0 + \beta_1 U_i$$
$$\text{logit}(P[Y_i(1) = 1|U_i]) = \alpha_0 + \alpha_1 U_i$$

- Consider the simpler case where there is no $\boldsymbol{X}_i$
- Our observed data provides estimates of $\text{p} = \mathbb{E}(W_i) = \mathbb{P}(W_i = 1)$, $\mu_{t,1} = \mathbb{E}[Y_i^{obs}|W_i = 1]$ and $\mu_{c,0} = \mathbb{E}[Y_i^{obs}|W_i = 0]$
- Given any value of $(q, \gamma_1, \beta_1, \alpha_1)$, we can solve the three equations to estimate $(\gamma_0, \beta_0, \alpha_0)$
- Then given the value of both $(q, \gamma_1, \beta_1, \alpha_1)$ and $(\hat{\gamma}_0, \hat{\beta}_0, \hat{\alpha}_0)$, we can estimate $\mu_{t,0} = \mathbb{E}[Y_i(1)|W_i = 0]$ and $\mu_{c,1} = \mathbb{E}[Y_i(0)|W_i = 1]$
- The average treatment effect will be

$$\tau_{\text{sp}} = \mu_{\text{t}} - \mu_{\text{c}} = p \cdot (\mu_{\text{t},1} - \mu_{\text{c},1}) + (1 - p) \cdot (\mu_{\text{t},0} - \mu_{\text{c},0}).$$

# Sensitivity Analysis

A more general approach (Rosenbaum book 2002)

Define $\pi_j = e(X_j, U_j)$ for a unit $j$. For a given $\Gamma$, assume

$$\frac{1}{\Gamma} \leq \frac{\pi_j(1-\pi_k)}{\pi_k(1-\pi_j)} \leq \Gamma \text{ all pairs of units } (j,k) \text{ with } X_j = X_k$$

Then we assess how the inference on causal effect change within the set for different $\Gamma$

- Tutorial (R package sensitivitymult):

  https://rosenbap.shinyapps.io/learnsenShiny/