Lecture 8
Pairwise randomized experiments

Outline

- pairwise randomized experiment
  - Fisher’s exact p-value
  - Neyman’s repeated sampling approach
  - Regression analysis
  - How to find strata / pairs?
  - R example
- Suggested reading: Imbens and Rubin Section 10.1 -10.6; Peng’s book Section 7.1-7.6

Pairwise randomized experiment

- Procedure:
  1. Create $J = N/2$ pairs of similar units
  2. Randomize treatment assignment within each pair

- Assignment probability

  A special case of stratified randomized experiment where $N(j) = 2$ and $N_t(j) = 1$

$$P(\boldsymbol{W} = \boldsymbol{w}|\boldsymbol{X}) = \begin{cases} \prod_{j=1}^{J} \binom{N(j)}{N_t(j)}^{-1} = 2^{-N/2} & \text{if } \sum_{i:B_i=j}^{N} w_i = 1 \text{ for } j = 1, \cdots, J \\ 0 & \text{otherwise} \end{cases}$$

The Children's television workshop experiment

[Ball, Bogatz, Rubin and Beaton, 1973.]

- The Educational Testing Service (ETS) wanted to evaluate *The Electric Company,* an American educational children's television series aimed at improving reading skills for young children

- Two sites, Yongstown, Ohio and Fresno, California where the show was not broadcast on local television, were selected to evaluate the effect of watching the show at school

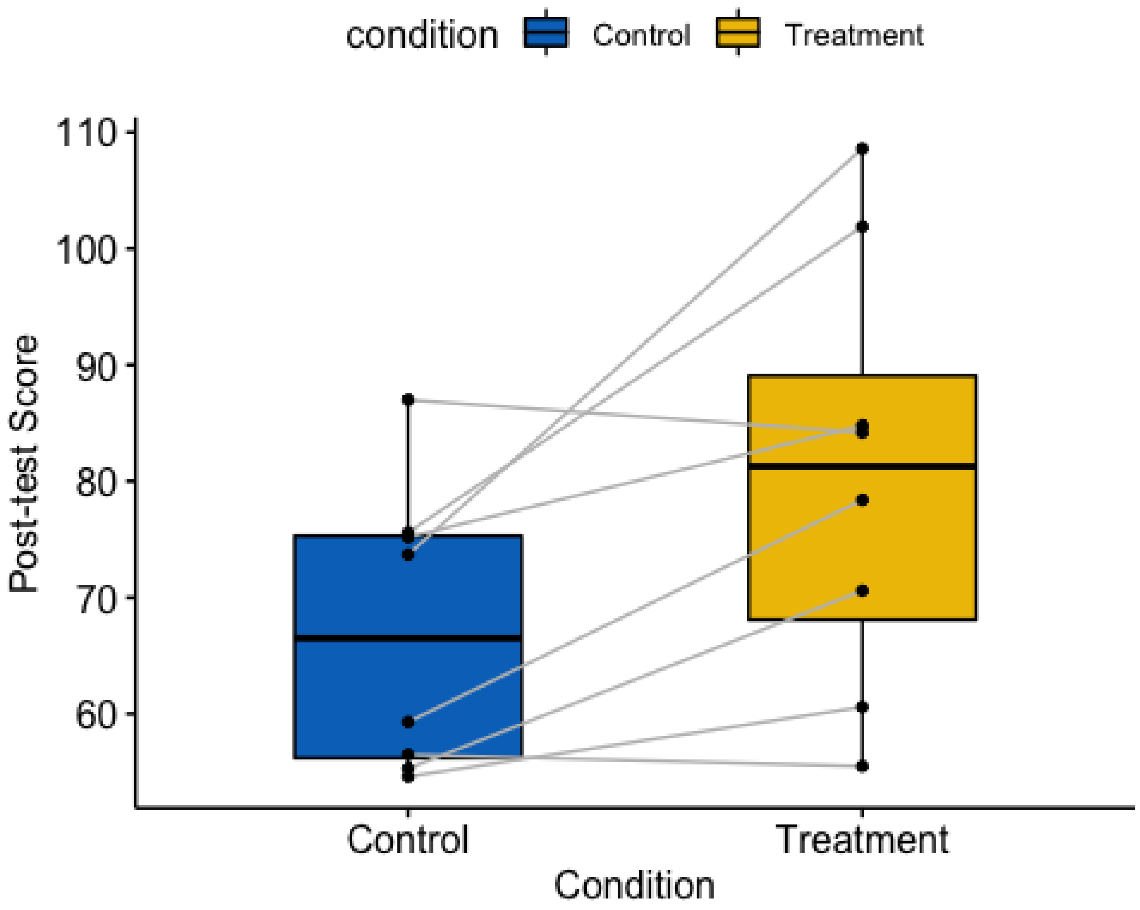
- Within each school, a pair of two classes are selected
  - One class randomly assigned to watch the show
  - Another class continue with regular reading curriculum

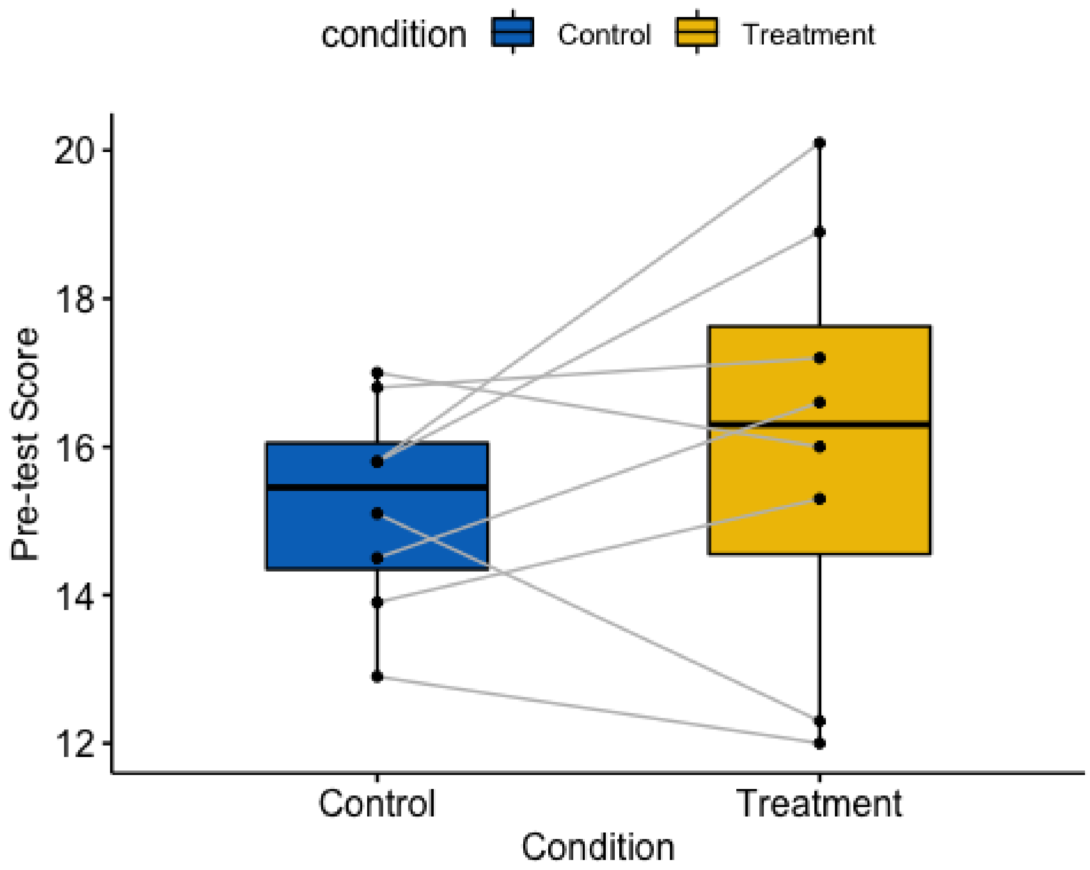
Data from Youngstown

Pair G_i	Treatment W_i	Pre-Test Score X_i	Post-Test Score Y_i^{obs}
1	0	12.9	54.6
1	1	12.0	60.6
2	0	15.1	56.5
2	1	12.3	55.5
3	0	16.8	75.2
3	1	17.2	84.8
4	0	15.8	75.6
4	1	18.9	101.9
5	0	13.9	55.3
5	1	15.3	70.6
6	0	14.5	59.3
6	1	16.6	78.4
7	0	17.0	87.0
7	1	16.0	84.2
8	0	15.8	73.7
8	1	20.1	108.6

- Two first-grade classes from each of eight schools participate in the experiment
- ETS performed reading ability tests to the kids both before the program started and after it finished.

Data from Youngstown





Some notations

Pair	Unit A					Unit B				
	$Y_{i,A}(0)$	$Y_{i,A}(1)$	$W_{i,A}$	$Y_{i,A}^{\text{obs}}$	$X_{i,A}$	$Y_{i,B}(0)$	$Y_{i,B}(1)$	$W_{i,B}$	$Y_{i,B}^{\text{obs}}$	$X_{i,B}$
1	54.6	?	0	54.6	12.9	?	60.6	1	60.6	12.0
2	56.5	?	0	56.5	15.1	?	55.5	1	55.5	13.9
3	75.2	?	0	75.2	16.8	?	84.8	1	84.8	17.2
4	76.6	?	0	75.6	15.8	?	101.9	1	101.9	18.9
5	55.3	?	0	55.3	13.9	?	70.6	1	70.6	15.3
6	59.3	?	0	59.3	14.5	?	78.4	1	78.4	16.6
7	87.0	?	0	87.0	17.0	?	84.2	1	84.2	16.0
8	73.7	?	0	73.7	15.8	?	108.6	1	108.6	20.1

- Average treatment effect within pair j

$$\tau^{\text{pair}}(j) = \frac{1}{2}\sum_{i:G_i=j}\big(Y_i(1) - Y_i(0)\big) = \frac{1}{2}\Big(\big(Y_{j,A}(1) - Y_{j,A}(0)\big) + \big(Y_{j,B}(1) - Y_{j,B}(0)\big)\Big).$$

- Observed outcomes for both treatment and control groups

$$Y_{j,c}^{\text{obs}} = \begin{cases} Y_{j,1}(0) & \text{if } W_{j1} = 0 \\ Y_{j,2}(0) & \text{if } W_{j2} = 0 \end{cases} \text{ and} Y_{j,t}^{\text{obs}} = \begin{cases} Y_{j,1}(1) & \text{if } W_{j1} = 1 \\ Y_{j,2}(1) & \text{if } W_{j2} = 1 \end{cases}$$

Fisher's exact p-value

- We still focus on the **Sharp null:** $H_0: Y_i(0) \equiv Y_i(1)$ for all $i = 1, \cdots, N$

- Choice of test statistics:
  - Average group mean differences across pairs

$$T^{\text{dif}} = \left| \frac{1}{J} \sum_{j=1}^{J} \left(Y_{j,t}^{\text{obs}} - Y_{j,c}^{\text{obs}} \right) \right| = \left| \bar{Y}_t^{\text{obs}} - \bar{Y}_c^{\text{obs}} \right|$$

    As each pair has exactly one treatment and one control
    - We don't need to consider different weights
    - No worry of Simpson's paradox

  - Rank statistics
    - Use population ranks: $T = |\overline{\text{rank}}(Y_t^{\text{obs}}) - \overline{\text{rank}}(Y_c^{\text{obs}})|$
    - Use within-pair ranks

$$T^{\text{rank,pair}} = \left| \frac{2}{N} \sum_{j=1}^{N/2} \left(\mathbf{1}_{Y_{j,1}^{\text{obs}} > Y_{j,0}^{\text{obs}}} - \mathbf{1}_{Y_{j,1}^{\text{obs}} < Y_{j,0}^{\text{obs}}} \right) \right|$$

Application to the television workshop data

- Fisher's exact p-values
  - Mean differences: $T = 13.4$, $\text{pvalue} = 0.031$
  - Rank mean differences: $T = 3.75$, $\text{pvalue} = 0.031$
  - Within-pair rank differences: $T = 0.5$, $\text{pvalue} = 0.29$

- Rank v.s. within-pair rank
  - Both can reduce the sensitivity to outliers
  - Using within-pair ranks can have more power when there is substantial variation in the level of the outcomes between pairs
  - Otherwise, using within-pair ranks loses power as it treats small within-pair differences (which may be due to random noises) equally with large within-pair differences

  - Using within-pair ranks is more appropriate for large, heterogenous population

Neyman’s repeated sampling approach

- Target: PATE or SATE $\tau = \sum_j \frac{N(j)}{N} \tau(j)$ where $\tau(j)$ is the PATE or SATE for strata j
- Point estimate:

$$\hat{\tau}^{\text{pair}}(j) = Y^{\text{obs}}_{j,t} - Y^{\text{obs}}_{j,c} \qquad \hat{\tau}^{\text{dif}} = \frac{1}{N/2} \sum_{j=1}^{N/2} \hat{\tau}^{\text{pair}}(j) = \overline{Y}^{\text{obs}}_{\text{t}} - \overline{Y}^{\text{obs}}_{\text{c}}$$

- $\mathbb{E}\left(\hat{\tau}^{\text{dif}}\right) = \tau$

$$\mathbb{E}\left(Y^{\text{obs}}_{j,t} - Y^{\text{obs}}_{j,c}\right) = \mathbb{E}\Big(W_{j1} Y_{j,1}(1) + W_{j2} Y_{j,2}(1) - (1 - W_{j1}) Y_{j,1}(0) - (1 - W_{j2}) Y_{j,2}(0)\Big) = \tau^{\text{pair}}(j)$$

- We can not estimate the within-pairs variances as there are only two units per pair
- Use the following empirical estimate of the uncertainty (paired t-test)

$$\hat{\mathbb{V}}^{\text{pair}}\left(\hat{\tau}^{\text{dif}}\right) = \frac{4}{N \cdot (N-2)} \cdot \sum_{j=1}^{N/2} \left(\hat{\tau}^{\text{pair}}(j) - \hat{\tau}^{\text{dif}}\right)^2$$

- Above estimate is conservative
  - $\hat{\tau}^{\text{pair}}(j)$ has mean $\tau^{\text{pair}}(j)$ instead of τ

$$\mathbb{E}\left[\hat{\mathbb{V}}^{\text{pair}}\left(\hat{\tau}^{\text{dif}}\right)\right] = \mathbb{V}_W(\hat{\tau}^{\text{dif}}) + \frac{4}{N \cdot (N-2)} \cdot \sum_{j=1}^{N/2} \left(\tau^{\text{pair}}(j) - \tau\right)^2$$

Application to the television workshop data

- Est. = 13.4, sd. = 4.6, 95% CI: $[4.3, 22.5]$
- As we have 8 pairs, Gaussian approximation is inaccurate and it's better to compare with a t-distribution with df = 7
- 95% CI comparing with t-distribution: $[2.5, 24.3]$
- If we treat the data as from completely randomized experiment, then sd. = 7.8

Pair	Outcome for Control Unit	Outcome for Treated Unit	Difference
1	54.6	60.6	6.0
2	56.5	55.5	−1.0
3	75.2	84.8	9.6
4	75.6	101.9	26.3
5	55.3	70.6	15.3
6	59.3	78.4	19.1
7	87.0	84.2	−2.8
8	73.7	108.6	34.9
Mean	67.2	80.6	13.4
(S.D.)	(12.2)	(18.6)	(13.1)

Linear regression

- We can not run separate linear regressions within each pair, as there are only 2 units per pair

How to build a reasonable regression framework?

- For each pair j, $Y_{j,k}(w) = Y_{j,k}(0) + \tau_{j,k}$ for $k = 1$ or 2
- We assume that

$$\mathbb{E}\big(Y_{j,k}(0)|\boldsymbol{X}\big) = \alpha_j + \boldsymbol{\beta}^T\boldsymbol{X}_{j,k}, \qquad \mathbb{E}\big(\tau_{j,k}|\boldsymbol{X}\big) = \tau + \boldsymbol{\gamma}^T\big(\boldsymbol{X}_{j,k} - \overline{\boldsymbol{X}}\big)$$

- Then

$$\mathbb{E}\big(Y_{j,k}(w)|\,\boldsymbol{X}_{jk}\big) = \alpha_j + \tau w + \boldsymbol{\beta}^T\boldsymbol{X}_{j,k} + w\boldsymbol{\gamma}^T\big(\boldsymbol{X}_{j,k} - \overline{\boldsymbol{X}}\big)$$

- Unconfoundedness property (also implicitly condition on pair indicators):

$$\big(\boldsymbol{Y}(0), \boldsymbol{Y}(1)\big) \perp \boldsymbol{W} \mid \boldsymbol{X}$$

- Then we have

$$\mathbb{E}\big(Y_{j,t}^{\mathrm{obs}} - Y_{j,c}^{\mathrm{obs}}|\boldsymbol{W} = \boldsymbol{w}, \boldsymbol{X} = \boldsymbol{x}\big) = \mathbb{E}\big(Y_{j,t}(1) - Y_{j,c}(0)|\boldsymbol{W} = \boldsymbol{w}, \boldsymbol{X} = \boldsymbol{x}\big)$$
$$= \mathbb{E}\big(Y_{j,t}(1) - Y_{j,c}(0)|\boldsymbol{X} = \boldsymbol{x}\big)$$

where $Y_{j,t}^{\mathrm{obs}}$ and $Y_{j,c}^{\mathrm{obs}}$ are observed responses for the treated and control unit in the jth pair

Linear regression

- We finally have the regression model:

$$\mathbb{E}\left(Y_{j,t}^{\text{obs}} - Y_{j,c}^{\text{obs}} | \boldsymbol{W} = \boldsymbol{w}, \boldsymbol{X} = \boldsymbol{x}\right) = \mathbb{E}\left(Y_{j,t}(1) - Y_{j,c}(0) | \boldsymbol{X} = \boldsymbol{x}\right)$$
$$= \tau + \boldsymbol{\gamma}^T\left(\boldsymbol{X}_{j,t} - \overline{\boldsymbol{X}}\right) + \boldsymbol{\beta}^T\left(\boldsymbol{X}_{j,t} - \boldsymbol{X}_{j,c}\right)$$
$$= \tau + \boldsymbol{\gamma}^T\left(\overline{\boldsymbol{X}}_j - \overline{\boldsymbol{X}}\right) + \left(\boldsymbol{\beta} + \frac{\boldsymbol{\gamma}}{2}\right)^T \left(\boldsymbol{X}_{j,t} - \boldsymbol{X}_{j,c}\right)$$

- τ is still the PATE
- We still implicitly condition on the pair indicators variables

- If $\boldsymbol{\gamma} = \boldsymbol{0}$, then $\mathbb{E}\left(Y_{j,t}^{\text{obs}} - Y_{j,c}^{\text{obs}} | \boldsymbol{W} = \boldsymbol{w}, \boldsymbol{X} = \boldsymbol{x}\right) = \tau + \boldsymbol{\beta}^T\left(\boldsymbol{X}_{j,t} - \boldsymbol{X}_{j,c}\right)$ we only need to include the covariates differences in the linear regression model

- We can assume homoscedastic errors in the linear regression even if $\mathbb{V}(Y_i(0)) \neq \mathbb{V}(Y_i(1))$
    - We assume the pairs are i.i.d.

How to perform stratification / pairing

- Implementation based on convenience

- Univariate blocking: discrete or discretized variable
- Multivariate blocking: Mahalanobis distance

$$D(\mathbf{X}_i, \mathbf{X}_j) = \sqrt{(\mathbf{X}_i - \mathbf{X}_j)^\top \widehat{\mathbb{V}(\mathbf{X})}^{-1} (\mathbf{X}_i - \mathbf{X}_j)}$$

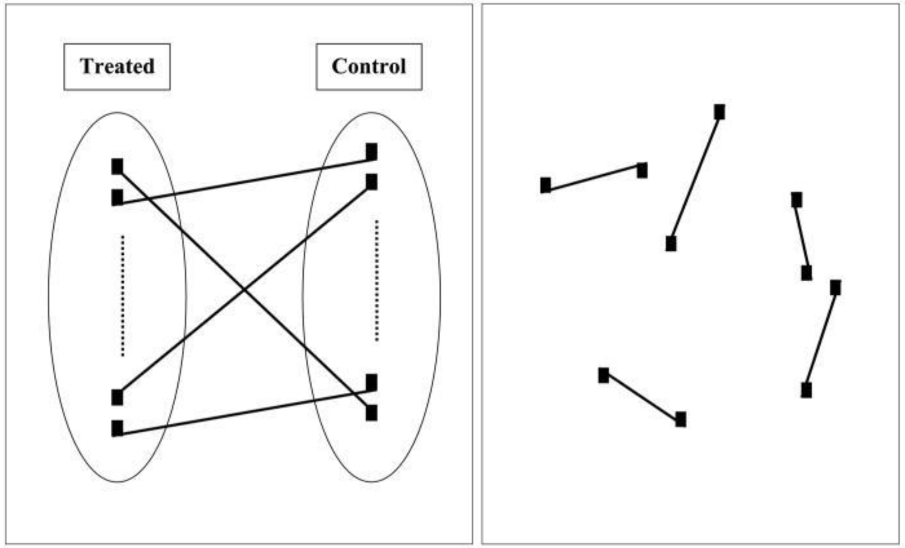
Greedy algorithms

- Matching: pair two units with the shortest distance, set them aside, and repeat
- Blocking: randomly choose one unit and choose N_j units with the shortest distances, set them aside, and repeat

But the resulting matches may not be optimal

Optimal matching

- D: $N \times N$ matrix of pairwise distance or a cost matrix
- Optimal matching
  - Binary $N \times N$ matching matrix: M with $M_{ij} \in \{0,1\}$
  - Optimization problem:
    $\min_M \sum_{i=1}^{N} \sum_{j=1}^{N} M_{ij} D_{ij}$ subject to $\sum_{i=1}^{N} M_{ij} = 1$ for all j
    where we set $D_{ii} = \infty$ for all i
  - M also need to be symmetric



Bipartite Matching Nonbipartite Matching

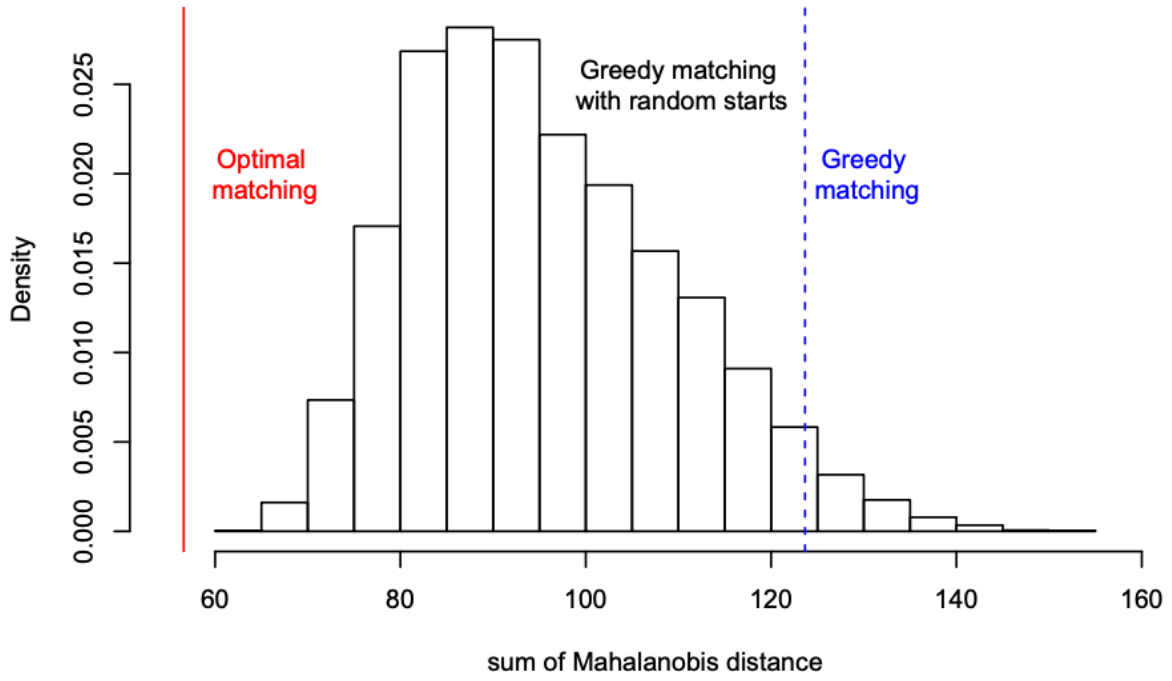
- Nonbipartite matching

- Computational cost $O(n^3)$
- Derigs' algorithm: implemented in the R package nbpMatching
  https://cran.r-project.org/web/packages/nbpMatching/

Example: evaluation of health insurance policy

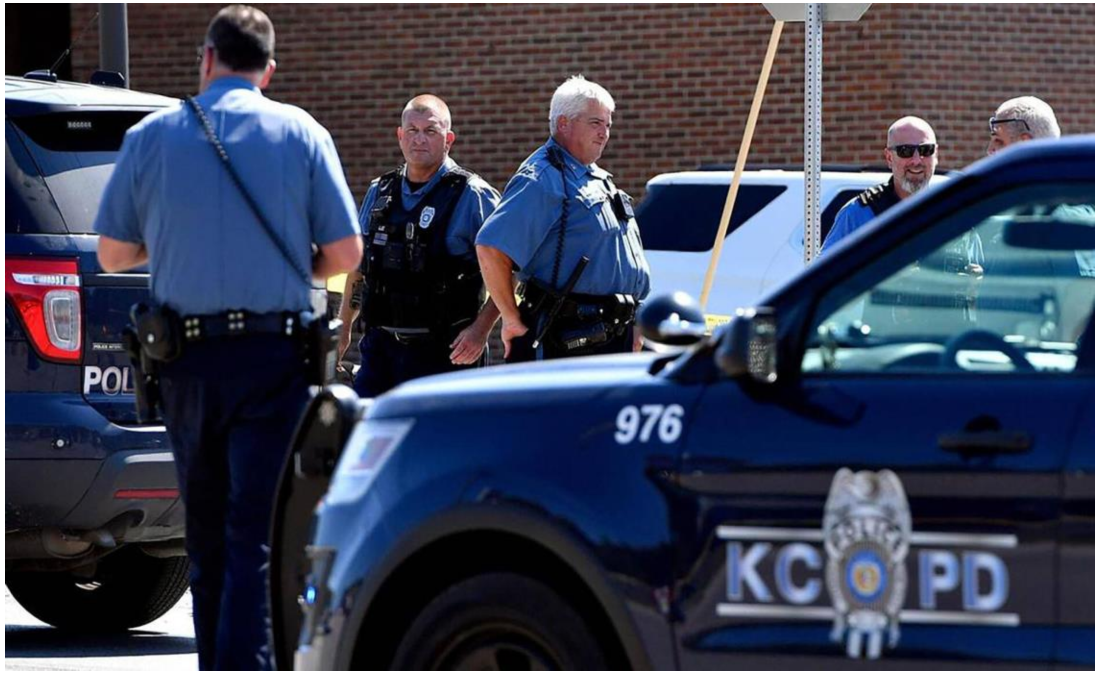
[Public policy for the poor? A randomised assessment of the Mexican universal health insurance programme. *The lancet*, *2009*.]

- Seguro Popular, a programme aimed to deliver health insurance, regular and preventive medical care, medicines, and health facilities to 50 million uninsured Mexicans
- Units: health clusters = predefined health facility catchment areas
- 4 pre-treatment cluster-average covariates: age, education, household size, household assets
- 100 clusters, 50 pairs



Case study: Kansas City Preventive Patrol Experiment

- A landmark experiment carried out between October 1, 1972, through September 30, 1973



Goal:

- Test for two fundamental hypotheses:
  1. **Visible Police Presence Deters Crime**: potential offenders would be less likely to commit crimes if they saw police patrols.
  2. **Police Presence Reduces Public Fear**: seeing police patrols would make the community feel safer.

Preventive patrol

police actively patrol an area in an attempt to prevent crime from occurring

Case study: Kansas City Preventive Patrol Experiment

Table 16:
PATROL IS THE MOST IMPORTANT FUNCTION IN THE POLICE DEPARTMENT

Total Responding = 178 0 = 1.94 S.D. = 1.05

Response of South Patrol Division Police Officers	
Strongly agree	42.2%
Moderately agree	32.8%
Slightly agree	1.7%
Slightly disagree	5.0%
Moderately disagree	0.6%
Strongly disagree	1.1%
No response	16.7%

Survey from the police

Experimental design

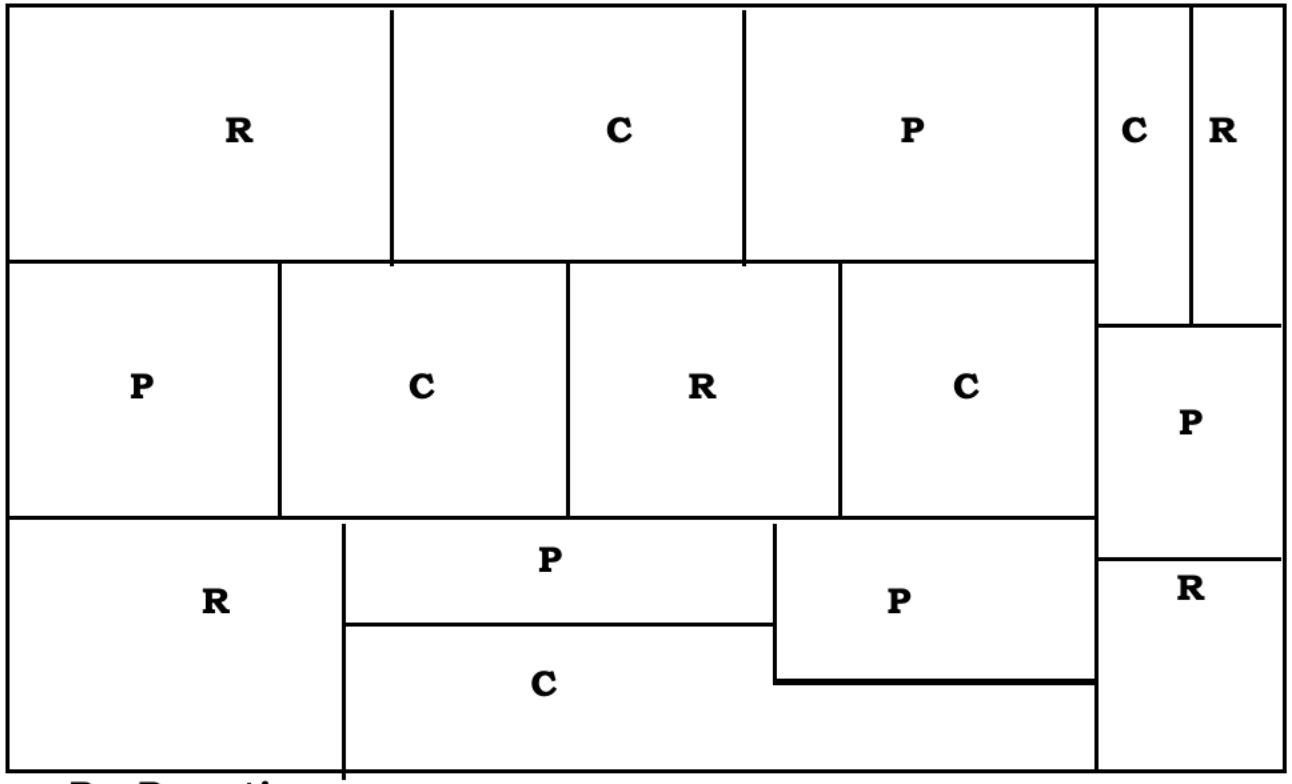
- Among South Patrol Division’s 24-beat area, nine beats were eliminated as unrepresentative of the city’s socioeconomic composition.

- The remaining 15 beats are computationally matched into 5 groups, 3 beats for each group

- Randomization within each group: randomly select one beat for each treatment level
  - Reactive Patrol(R): Police cars were removed from these beats. Officers only responded to calls for service.
  - Standard Patrol (C): These beats acted as the control group, with policing continuing as usual.
  - Proactive Patrol (P): Police patrols were significantly increased in these beats.

- It was agreed that if a noticeable increase in crime occurred within a reactive beat, the experiment would be suspended.
- Additional training to the police that encourage them to adhere to the treatment assignment

Experimental design and outcome



P = Proactive
C = Control
R = Reactive

Outcome measured

- Crime rates
- Response times
- Community attitudes toward the police

- Data are collected from community surveys, interviews, recorded observations and departmental data

Analysis result

no significant differences in the level of crime, citizens' attitudes toward police services, citizens' fear of crime, police response time, or citizens' satisfaction with police response time.

Summary report available at: https://www.policinginstitute.org/wp-content/uploads/2015/07/Kelling-et-al.-1974-THE-KANSAS-CITY-PREVENTIVE-PATROL-EXPERIMENT.pdf

Table 2: DEPARTMENTAL REPORTED CRIME

Crime Type	Overall P	R,C	R,P	C,P
Robbery - Inside		R=C	R=P	C=P
Robbery - Outside		R=C	R=P	C=P
Common Assault		R=C	R=P	C=P
Aggravated Assault		R=C	R=P	C=P
Larceny - Purse Snatch		R=C	R=P	C=P
Rape		R=C	R=P	C=P
Other Sex Crimes	$.01<p<.025$	R>C	R=P	C=P
Homicide		R=C	R=P	C=P
Residence Burglary		R=C	R=P	C=P
Non-Residence Burglary		R=C	R=P	C=P
Auto Theft		R=C	R=P	C=P
Vandalism		R=C	R=P	C=P
Larceny - Auto Accessory		R=C	R=P	C=P
Larceny - Theft from Auto		R=C	R=P	C=P
Larceny - Bicycle		R=C	R=P	C=P
Larceny - Shoplift		R=C	R=P	C=P
Larceny - Theft from Bldg.		R=C	R=P	C=P

Comments on the analysis result

What can be the potential drawbacks of the experimental design and analysis?

- Data analyzed by two-sample testing, not as from paired randomized experiment, so statistical test can be conservative
- Sample size is small
- Short term effect may be small

- Non-compliance → Police presence are kept monitored during the experiment
  - However, the study did not collect data on the amount of preventive patrol in each condition (Weisburd et. al. 2023)
- Spill-over effect → Assessed by evaluating correlation between nearby beats to indicate no spill-over effect
- The randomization is questioned (Weisburd et. al. 2023): four R beats are on the corner of the region