Lecture 13
Matching methods

Outline

- Outcome regression V.S. Matching
- Find matched sets
  - Matching metrics and algorithms
  - Check covariate balancing
- Estimate ATT after matching
  - Bias adjustment
- Suggested reading: Imbens and Rubin book Chapter 15 & 18, Peng’s book Chapter 15

Causal estimand

- If we treat the units as sampled from a population
  - Population average treatment effect: $\text{PATE} = \text{ATE} = \mathbb{E}(Y_i(1) - Y_i(0))$
  - Average treatment effect for the treated: $\text{PATT} = \text{ATT} = \mathbb{E}(Y_i(1) - Y_i(0) \mid W_i = 1)$
  - Average treatment effect for the control: $\text{ATC} = \mathbb{E}(Y_i(1) - Y_i(0) \mid W_i = 0)$

$$\text{ATE} = P(W_i = 1) \times \text{ATT} + P(W_i = 0) \times \text{ATC}$$

- In randomized experiments, ATE is equivalent to ATT, because treatment and control groups are comparable in expectation

- In observational studies, we can be interested in ATT
  - Many dataset can have a modest number of treated units, but a relatively large pool of possible controls
  - Treated units are more well defined
  - Control units may include units that never have a chance to receive treatment

Outcome regression estimator

- The outcome regression estimator is the same as in conditional randomized experiment
- Under unconfoundedness assumption

$$\tau = \mathbb{E}\left(\mathbb{E}\left(Y_i^{\text{obs}} \middle| \boldsymbol{X}_i, W_i = 1\right) - \mathbb{E}\left(Y_i^{\text{obs}} \middle| \boldsymbol{X}_i, W_i = 0\right)\right)$$

- Define the conditional expectations

$$\mu_w(\boldsymbol{x}) = \mathbb{E}\left(Y_i^{\text{obs}} \middle| \boldsymbol{X}_i = \boldsymbol{x}, W_i = w\right) = \mathbb{E}(Y_i(w) | \boldsymbol{X}_i = \boldsymbol{x})$$

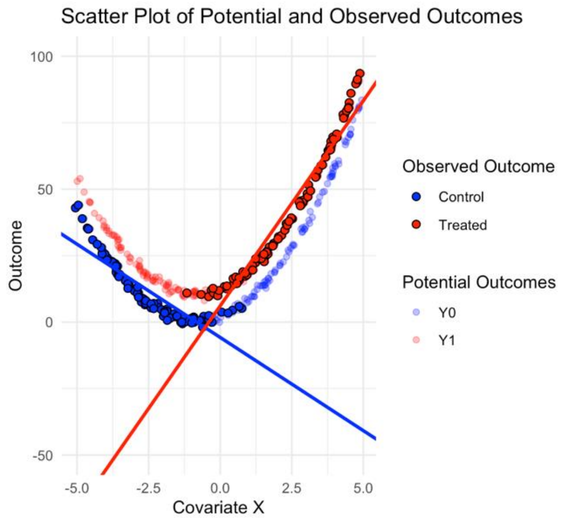
model assumptions on the potential outcomes

- We can estimate the conditional expectations via a regression model and obtain $\hat{\mu}_w(\boldsymbol{x})$
  - Regress Y_i^{obs} on $\boldsymbol{X}_i$ on the treated units and control units separately

- Estimator for the ATE: implement unobserved potential outcome by regression estimates

$$\hat{\tau}_{\text{reg}} = \frac{1}{N}\left\{\sum_{i=1}^{N} W_i \left(Y_i^{\text{obs}} - \hat{\mu}_0(\boldsymbol{X}_i)\right) + (1 - W_i)\left(\hat{\mu}_1(\boldsymbol{X}_i) - Y_i^{\text{obs}}\right)\right\}$$

Regression estimator V.S. Matching



- Estimator for the ATT from regression

$$\hat{\tau}_{\text{reg}} = \frac{1}{N_t} \sum_{i=1}^{N} W_i \left(Y_i^{\text{obs}} - \hat{\mu}_0(\boldsymbol{X}_i) \right)$$

  - Model-based imputation of unobserved potential outcomes
  - Drawbacks:
    - biased imputation if model is wrong
    - If the imbalance of the covariates between the two groups is large, the model-based results heavily relies on extrapolation in the region with little overlap, which is sensitive to the model specification assumption

- Matching: nonparametric imputation

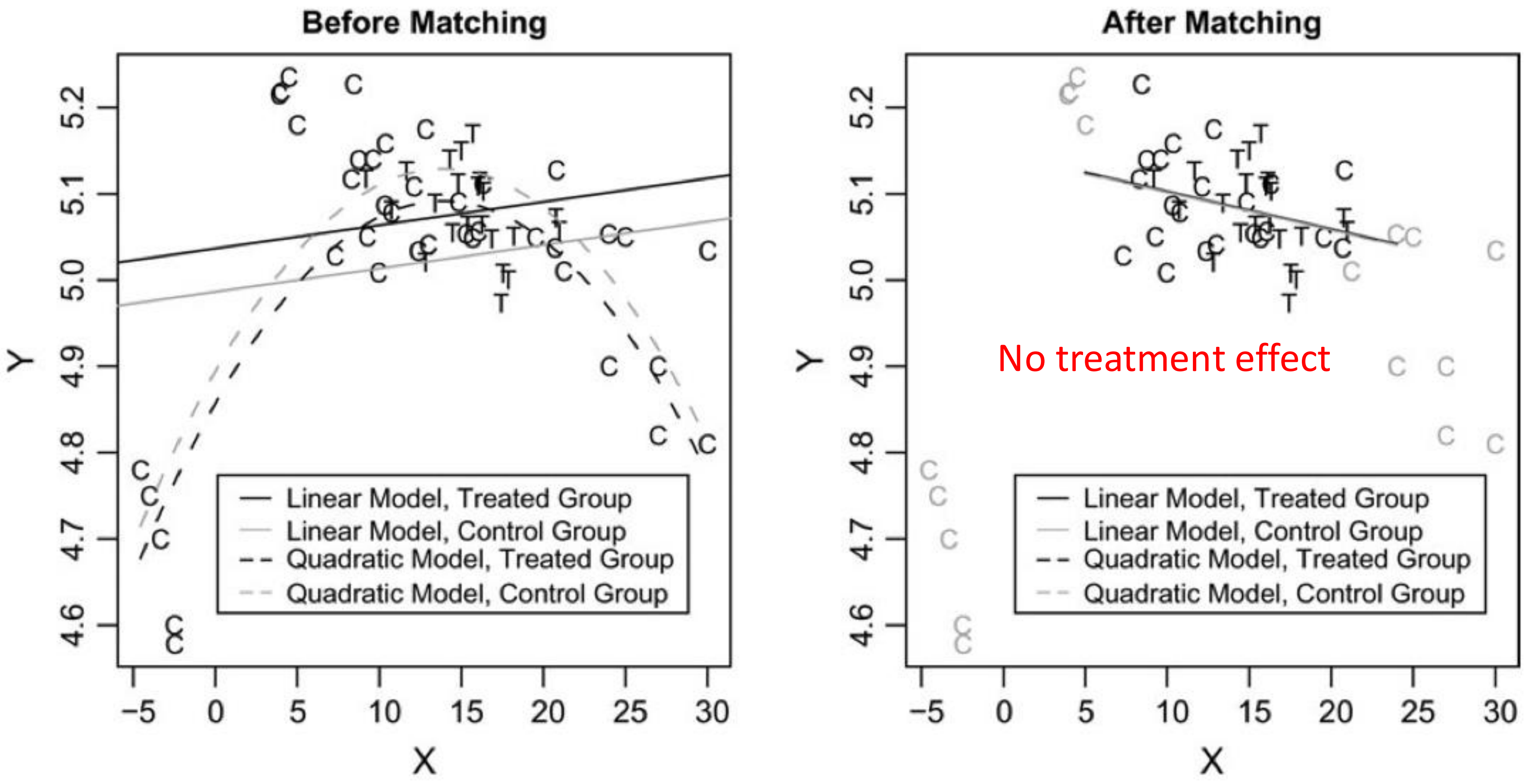
$$\hat{\tau}_{\text{match}} = \frac{1}{N_t} \sum_{i=1}^{N} W_i \left(Y_i^{\text{obs}} - \frac{1}{|\mathcal{M}_i^c|} \sum_{i' \in \mathcal{M}_i^c} Y_{i'}^{\text{obs}} \right)$$

  - $\mathcal{M}_i^c$: matched set of controls for treated unit i

A simulation data example

[Matching as nonparametric preprocessing for reducing model dependence in parametric causal inference. *Political analysis*, 2007]

- Linear regression: positive treatment effect
- Quadratic regression: negative treatment effect
- Both are wrong!!



- At the two extreme tails of X, there are no treatment units at all

How to find matched sets?

- Matching with replacement v.s. matching without replacement
  - Whether we restrict each control to match with at most one treated unit or not
  - Matching without replacement: harder matching algorithm but easier statistical inference

- Exact match: perfect covariate balance $\boldsymbol{X}_i$ for the matched control(s) are the same as the treated unit
  - Infeasible when covariate is continuous / many covariates

- Coarsened exact matching (Lacus et al. 2011 Political Anal.)
  - discretize covariates so that you can perform exact match

- Matching based on a distance
  - Define a distance measure for any two units: $D(\boldsymbol{X}_i, \boldsymbol{X}_j)$
  - Aim to make units within matched sets as close as possible

Matching based on a distance

- Mahalanobis metric matching

$$D(\mathbf{X}_i, \mathbf{X}_j) = \sqrt{(\mathbf{X}_i - \mathbf{X}_j)^\top \widehat{\mathbb{V}(\mathbf{X})}^{-1} (\mathbf{X}_i - \mathbf{X}_j)}$$

$\widehat{\mathbb{V}(\boldsymbol{X})} = \frac{N_t\hat{\Sigma}_t + N_c\hat{\Sigma}_c}{N_t + N_c}$, $\hat{\Sigma}_t$ and $\hat{\Sigma}_c$ are sample covariance matrices for the treated and control

- Propensity score matching

$$D(\boldsymbol{X}_i, \boldsymbol{X}_j) = \left|\ln\left(\frac{\hat{e}(\boldsymbol{X}_i)}{1 - \hat{e}(\boldsymbol{X}_i)}\right) - \ln\left(\frac{\hat{e}(\boldsymbol{X}_j)}{1 - \hat{e}(\boldsymbol{X}_j)}\right)\right|$$

- Hybrid matching methods
  - Ensure exact matching in some key covariates: sex
  - First stratify units by key covariates, match within each strata using distance-based matching

Matching based on a distance

Nearest-neighbor (NN) matching:

- Define $\mathcal{M}_i^c$ as the set of indices of M closest control units

$$\mathcal{M}_i^c = \left\{ j: W_j = 0, \sum_{l|W_l=0} 1_{\{D(\boldsymbol{X}_i, \boldsymbol{X}_j) \leq D(\boldsymbol{X}_i, \boldsymbol{X}_l)\}} \leq M \right\}$$

- Matching with replacement

Greedy algorithm

- Define an order of the treated units
- Match M control units with the shortest distance, set them aside, and repeat
- match most difficult units first: order treated units in a descending order of $\hat{e}(\boldsymbol{X}_i)$

Optimal matching

- $D: N_t \times N_c$ bipartite matrix of pairwise distance or a cost matrix
- Select $N_t M$ elements of D such that there is only M elements in each row and at most one element in each column and the sum of pairwise distances is minimized

Optimal matching

- D: $N_t \times N_c$ matrix of pairwise distance or a cost matrix
- Select $N_t M$ elements of D such that there is only M element in each row and at most one element in each column and the sum of pairwise distances is minimized

- Linear Sum Assignment Problem (LSAP)
  - Binary $N_t \times N_c$ matching matrix: S with $S_{ij} \in \{0,1\}$
  - Optimization problem

$$\min_S \sum_{i=1}^{N_t} \sum_{j=1}^{N_c} S_{ij} D_{ij} \quad \text{subject to} \sum_{i=1}^{N_t} S_{ij} \leq 1, \sum_{j=1}^{N_c} S_{ij} = M$$

  - can apply the Hungarian algorithm

A simple illustrative example

- Consider 7 units

- Matching based on the linearized estimated propensity score

$$\hat{l}(\boldsymbol{X}_i) = \ln\left(\frac{\hat{e}(\boldsymbol{X}_i)}{1 - \hat{e}(\boldsymbol{X}_i)}\right)$$

- Treated unit 1 matched with control unit 5
- Treated unit 2 matched with control unit 3

- NN, greedy algorithm and optimal matching result in the same matched sets here

Unit	W_i	$\hat{e}(X_i)$	$\hat{\ell}(X_i)$
1	1	0.577	0.310
2	1	0.032	−3.398
3	0	0.136	−1.846
4	0	0.003	−5.913
5	0	0.310	−0.798
6	0	0.000	−9.424
7	0	0.262	−1.033

Further restrictions on the matched sets

- Rejecting matches of poor quality
  - For some units, even the closest match may not be close enough
  - Drop treated units if it's hard to find a good match. E.x., drop i if

$$D(\boldsymbol{X}_i, \boldsymbol{X}_j) > d_{\max} = 0.1$$

    - Often eliminate only treated units with propensity score very close to 1

- How to determine M?
  - $M = 1$
  - Matching with Caliper: controls that are outside of some distance (caliper) of a treated unit are not allowed to be matched with the treated units.
    - Keep all controls j satisfying $D(\boldsymbol{X}_i, \boldsymbol{X}_j) \leq d_{\text{cal}}$
    - Can use greedy algorithm
    - Optimal matching: define $D_{ij} = \infty$ if $D_{ij} > d_{\text{cal}}$
  - M increases with sample size
  - Smaller M, smaller bias but larger variance; larger M, larger bias but smaller variance

Check covariate balancing after matching

- Statistics we can use to assess the balancing of a particular covariate
  - **Standardized mean difference** (also called the normalized difference, not the t-statistics)

$$\Delta_{ct} = \frac{\frac{1}{N_t}\sum_{i=1}^{N} W_i \left(X_{ik} - \frac{1}{|\mathcal{M}_i^c|}\sum_{i' \in \mathcal{M}_i^c} X_{i'k} \right)}{\sqrt{s_t^2}}$$

    May compare Δ_{ct} with 0.1
    - Before matching, we may calculate the denominator of Standardized mean difference as $\sqrt{(s_t^2 + s_c^2)/2}$
  - **Log ratio of the sample variances** $\Gamma_{ct} = \ln(s_t) - \ln(s_c)$
  - Comparing the distribution function in the treated group and control group
    - Empirical cdf: $\hat{F}_c(x) = \frac{1}{N_c}\sum_{i:W_i=0} \mathbf{1}_{X_i \le x}, \quad \text{and} \quad \hat{F}_t(x) = \frac{1}{N_t}\sum_{i:W_i=1} \mathbf{1}_{X_i \le x}$
    - Proportion of treated units outside of the 2.5% and 97.5% quantiles of the control distribution

$$\hat{\pi}_t^{0.05} = \left(1 - \left(\hat{F}_t\left(\hat{F}_c^{-1}(0.975)\right)\right) + \hat{F}_t\left(\hat{F}_c^{-1}(0.025)\right)\right)$$

Love plot

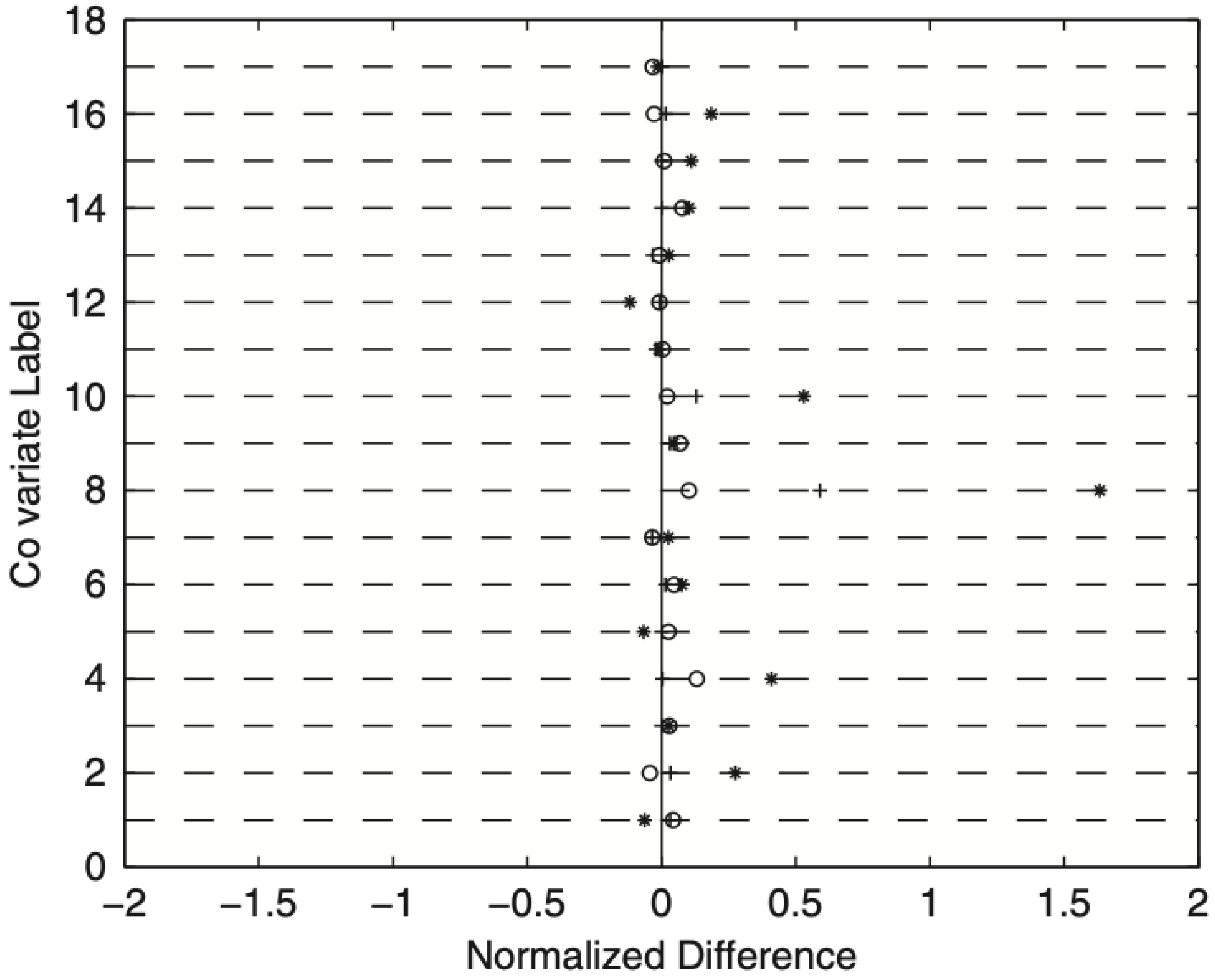


Figure 15.2. Covariate balance before (*) and after (+) lps and after Mahalanobis (o) matching, for the Reinisch barbiturate data

How to estimate ATT after matching

- Unless exact matching, under unconfoundedness, the probability of assignment to the treatment is only approximated the same within each matched set

- In practice, one may **ignore** the potential bias, and analyze the datasets as from a pairwise randomized experiment

$$\hat{\tau}_i^{\text{match}} = Y_i^{\text{obs}} - Y_{m_i^c}^{\text{obs}}, \qquad \hat{\tau}_{\text{t}}^{\text{match}} = \frac{1}{N_{\text{t}}} \sum_{i:W_i=1} \hat{\tau}_i^{\text{match}}$$

$$\hat{\mathbb{V}}\left(\hat{\tau}_{\text{t}}^{\text{match}}\right) = \frac{1}{N_t(N_t - 1)} \sum_{i:W_i=1} \left(Y_i^{\text{obs}} - Y_{m_i^c}^{\text{obs}} - \hat{\tau}_{\text{t}}^{\text{match}}\right)^2$$

- Another approach is to apply outcome regression on the matched dataset
  - Treat matching as a pre-processing step to improve covariate balancing in the dataset
  - Reduce bias in matching
  - Or we can use regression to only adjust for the potential biases (see later)

The minimum wage data

- An influential study by Card and Krueger (1995)
- The goal is to evaluate the effect of raising the state minimum wage in Ney Jersey in 1993

- They collected data on employment at fast-food restaurants in Ney Jersey (treated group) and in neighboring state of Pennsylvania (control group)

- Each unit is a restaurant
- Pre-treatment covariates: initial number of employees, starting wage, average time until first raise, identity of the chain
- Outcome: number of employees after the raise in the minimum wage

The minimum wage data

Table 18.1. ***The Card-Krueger New Jersey and Pennsylvania Minimum Wage Data***

	$(N = 347)$		$(N_c = 68)$ (controls)		$(N_t = 279)$ (treated)			
	Mean	(S.D.)	Mean	(S.D.)	Mean	(S.D.)	Nor Dif	Log Ratio of STD
initial empl	17.84	(9.62)	20.17	(11.96)	17.27	(8.89)	−0.28	−0.30
burger king	0.42	(0.49)	0.43	(0.50)	0.42	(0.49)	−0.02	−0.01
kfc	0.19	(0.40)	0.13	(0.34)	0.21	(0.41)	0.20	0.17
roys	0.25	(0.43)	0.25	(0.44)	0.25	(0.43)	0.00	−0.00
wendys	0.14	(0.35)	0.19	(0.40)	0.13	(0.33)	−0.18	−0.18
initial wage	4.61	(0.34)	4.62	(0.35)	4.60	(0.34)	−0.05	−0.02
time until raise	17.96	(11.01)	19.05	(13.46)	17.69	(10.34)	−0.11	−0.26
pscore	0.80	(0.05)	0.79	(0.06)	0.81	(0.04)	0.28	−0.35
final empl	17.37	(8.39)	17.54	(7.73)	17.32	(8.55)		

The minimum wage data

Estimated propensity score model:
Higher initial employment, lower propensity score

$$\hat{l}(\boldsymbol{X}_i) = 1.93 - 0.03 \times \text{ initial empl}$$

Table 18.2. ***Estimated Parameters of Propensity Score for the Card-Krueger New Jersey and Pennsylvania Minimum Wage Data***

Variable	Est	$(\widehat{\text{s. e.}})$	t-Stat
Intercept	1.93	(0.14)	14.05
Linear terms			
`initial empl`	–0.03	(0.01)	–2.17

The minimum wage data on 20 units

Unit i	State W_i	chain X_{i1}	initial empl X_{i2}	final empl Y_i^{obs}
1	NJ	BK	22.5	40.0
2	NJ	KFC	14.0	12.5
3	NJ	BK	37.5	20.0
4	NJ	KFC	9.0	3.5
5	NJ	KFC	8.0	5.5
6	PA	BK	10.5	15.0
7	PA	KFC	13.8	17.0
8	PA	KFC	8.5	10.5
9	PA	BK	25.5	18.5
10	PA	BK	17.0	12.5
11	PA	BK	20.0	19.5
12	PA	BK	13.5	21.0
13	PA	BK	19.0	11.0
14	PA	BK	12.0	17.0
15	PA	BK	32.5	22.5
16	PA	BK	16.0	20.0
17	PA	KFC	11.0	14.0
18	PA	KFC	4.5	6.5
19	PA	BK	12.5	31.5
20	PA	BK	8.0	8.0

- Matching order:
  if we rank based on $\hat{e}(\boldsymbol{X}_i)$: 5, 4, 2, 1, 3
- Matching metric:
  - Only based on $\hat{l}(\boldsymbol{X}_i)$: 20, 8, 7, 11, 15
  - If we want exact match on the chain brand
    5 <-> 8, 4 <->17, 2 <->7, 1<-> 11, 3 <-> 15
  - If we want to match on Mahalanobis distance, can code the restaurant brand by 0/1 indicators, then 5 <-> 20, 4 <-> 8

The minimum wage data on 20 units

i	m_i^c	Y_i^{obs}	$Y_{m_i^c}^{obs}$	$\hat{\tau}_i^{match}$
1	11	40.0	19.5	20.5
2	7	12.5	17	−4.5
3	15	20.0	22.5	−2.5
4	8	3.5	10.5	−7
5	20	5.5	8.0	−2.5
$\hat{\tau}_t^{match}$				+0.8
$\hat{\mathbb{V}}\left(\hat{\tau}_t^{match}\right)$				5.0^2

i	m_i^c	Y_i^{obs}	$Y_{m_i^c}^{obs}$	$\hat{\tau}_i^{match}$
1	11	40.0	19.5	20.5
2	7	12.5	17.0	−4.5
3	15	20.0	22.5	−2.5
4	17	3.5	14	-10.5
5	8	5.5	10.5	-5
$\hat{\tau}_t^{match}$				−0.4
				5.4^2

The bias of matching estimators (1-1 matching)

- Individual treatment effect is estimated with a bias due to matching discrepancy

$$\mathbb{E}_{\text{sp}}\left[\hat{\tau}_i^{\text{match}} \middle| W_i = 1, X_i, X_{m_i^c}\right] = \mathbb{E}_{\text{sp}}\left[Y_i(1) - Y_{m_i^c}(0) \middle| X_i, X_{m_i^c}\right] = \mu_{\text{t}}(X_i) - \mu_{\text{c}}(X_{m_i^c})$$

$$= \tau(X_i) + (\mu_{\text{c}}(X_i) - \mu_{\text{c}}(X_{m_i^c})).$$

We refer to the last term of this expression,

$$B_i = \mu_{\text{c}}(X_i) - \mu_{\text{c}}(X_{m_i^c}),$$

as the *unit-level bias* of the matching estimator.

- If we can have estimates of B_i, then we can potentially correct for the biases
- We can obtain the estimates of B_i by outcome regression: only need an estimate $\hat{\mu}_0(\boldsymbol{X}_i)$

$$\hat{\tau}_i^{\text{match}} = Y_i^{\text{obs}} - (Y_{m_i^c}^{\text{obs}} + \hat{B}_i)$$

Three types of regression

- Regression on the differences

$$Y_i^{\mathrm{obs}} - Y_{\mathrm{m}_i^c}^{\mathrm{obs}} = \tau + \left(X_i - X_{\mathrm{m}_i^c}\right)\beta_d + \nu_i = \tau + D_i\beta_d + \nu_i$$
$$\hat{B}_i = \left(X_i - X_{\mathrm{m}_i^c}\right)\hat{\beta}_d$$

- Regression only on the matched control

$$Y_{\mathrm{m}_i^c} = \alpha_{\mathrm{c}} + X_{\mathrm{m}_i^c}\beta_{\mathrm{c}} + \nu_{ci}$$
$$\hat{B}_i = (X_i - X_{\mathrm{m}_i^c})\hat{\beta}_{\mathrm{c}}$$

- Regression on both the treated and the matched controls (pooled sample)

$$\tilde{Y}_i = \alpha_p + \tau_p \cdot \tilde{W}_i + \tilde{X}_i\beta_p + \nu_i$$
$$\hat{B}_i = (X_i - X_{\mathrm{m}_i^c})\hat{\beta}_p$$

- These methods differ in their robustness to model assumptions and efficiency

Results on the 20 units

	Difference Regression (Approach #1)	Control Regression (Approach #2)	Pooled Regression (Approach #3)
Regression coefficients			
Intercept	−1.30	4.21	12.01
Treatment indicator	–	–	1.63
Restaurant chain	−1.20	2.65	−7.32
Initial employment	1.43	0.62	0.39

- Different regression methods differ a lot because small sample size
- In real data, they are typically similar