Lecture 15
Doubly Robust Estimator

Outline

- Doubly robust estimator
  - Interpretation
  - Comparison with IPW and outcome regression
  - A well-known simulation study
- Suggested Reading: Peng’s book Chapter 12

Two ways to estimate the ATE

- Outcome regression

$$\begin{aligned}\tau(\boldsymbol{x}) &= \mathbb{E}(Y_i(1) \mid \boldsymbol{X}_i = \boldsymbol{x}, W_i = 1) - \mathbb{E}(Y_i(0) \mid \boldsymbol{X}_i = \boldsymbol{x}, W_i = 0) \\ &= \mathbb{E}\left(Y_i^{\text{obs}} \middle| \boldsymbol{X}_i = \boldsymbol{x}, W_i = 1\right) - \mathbb{E}\left(Y_i^{\text{obs}} \middle| \boldsymbol{X}_i = \boldsymbol{x}, W_i = 0\right) \\ &= \mu_1(\boldsymbol{x}) - \mu_0(\boldsymbol{x})\end{aligned}$$

  - relies on a correctly specified model for the outcomes depending on $\boldsymbol{X}_i$

$$\hat{\tau}^{\text{reg}} = \frac{1}{N}\sum_{i=1}^{N} \hat{\mu}_1(\boldsymbol{X}_i) - \frac{1}{N}\sum_{i=1}^{N} \hat{\mu}_0(\boldsymbol{X}_i)$$

- IPW / Matching

$$\tau = \mathbb{E}\left(\frac{Y_i^{\text{obs}} W_i}{e(\boldsymbol{X}_i)}\right) - \mathbb{E}\left(\frac{Y_i^{\text{obs}} \cdot (1 - W_i)}{1 - e(\boldsymbol{X}_i)}\right)$$

  - relies on a correctly specified model for the propensity score

$$\hat{\tau}^{\text{IPW}} = \frac{1}{N}\sum_{i=1}^{N} \left\{\frac{W_i Y_i^{\text{obs}}}{\hat{e}(\boldsymbol{X}_i)} - \frac{(1 - W_i) Y_i^{\text{obs}}}{1 - \hat{e}(\boldsymbol{X}_i)}\right\}$$

Doubly robust estimator

- Can we provide a good estimate if either model is correct?
- Doubly robust estimator: provide a good estimate of ATE when either the outcome or the propensity score model is correct
- Define

$$f(1, \boldsymbol{X}_i, Y_i^{\text{obs}}) = \frac{Y_i^{\text{obs}} 1_{W_i=1}}{\tilde{e}(\boldsymbol{X}_i)} - \frac{1_{W_i=1} - \tilde{e}(\boldsymbol{X}_i)}{\tilde{e}(\boldsymbol{X}_i)} \tilde{\mu}_1(\boldsymbol{X}_i)$$

$$f(0, \boldsymbol{X}_i, Y_i^{\text{obs}}) = \frac{Y_i^{\text{obs}} 1_{W_i=0}}{1 - \tilde{e}(\boldsymbol{X}_i)} - \frac{1_{W_i=0} - (1 - \tilde{e}(\boldsymbol{X}_i))}{1 - \tilde{e}(\boldsymbol{X}_i)} \tilde{\mu}_0(\boldsymbol{X}_i)$$

- $\tilde{e}(\boldsymbol{X}_i)$, $\tilde{\mu}_w(\boldsymbol{X}_i)$: our working models (model under our model assumption)
- $e(\boldsymbol{X}_i)$, $\mu_w(\boldsymbol{X}_i)$: true model that we don’t know
  - If we correctly specify the propensity score model, then $\tilde{e}(\boldsymbol{X}_i) = e(\boldsymbol{X}_i)$
  - If we correctly specify the outcome model, then $\tilde{\mu}_w(\boldsymbol{X}_i) = \mu_w(\boldsymbol{X}_i)$

Interpretation

$$f(1, \boldsymbol{X}_i, Y_i^{\text{obs}}) = \underbrace{\frac{Y_i^{\text{obs}} 1_{W_i=1}}{\tilde{e}(\boldsymbol{X}_i)}} - \underbrace{\frac{1_{W_i=1} - \tilde{e}(\boldsymbol{X}_i)}{\tilde{e}(\boldsymbol{X}_i)} \tilde{\mu}_1(\boldsymbol{X}_i)}$$

IPW estimate of $\mathbb{E}(Y_i(1)|\ \boldsymbol{X}_i\ )$

Adjust for bias if the propensity score model is incorrect
(if PS model is correct, then this part has expectation 0)

An equivalent expression:

$$f(1, \boldsymbol{X}_i, Y_i^{\text{obs}}) = \underbrace{\tilde{\mu}_1(\boldsymbol{X}_i)} + \underbrace{\frac{1_{W_i=1}}{\tilde{e}(\boldsymbol{X}_i)} (\overset{Y_i(1)}{\overset{\shortparallel}{Y_i^{\text{obs}}}} - \tilde{\mu}_1(\boldsymbol{X}_i))}$$

Outcome regression estimate of $\mathbb{E}(Y_i(1)|\ \boldsymbol{X}_i\ )$

Adjust for bias if the outcome regression model is incorrect
(if PS model is correct, then this part has expectation 0)

Doubly robust property

- Double robust property

$$\mathbb{E}\left[f(1, \boldsymbol{X}_i, Y_i^{\text{obs}}) \mid \boldsymbol{X}_i\right] = \frac{\left(\mu_1(\boldsymbol{X}_i) - \tilde{\mu}_1(\boldsymbol{X}_i)\right)\left(e(\boldsymbol{X}_i) - \tilde{e}(\boldsymbol{X}_i)\right)}{\tilde{e}(\boldsymbol{X}_i)} + \mu_1(\boldsymbol{X}_i)$$

$$\mathbb{E}\left[f(0, \boldsymbol{X}_i, Y_i^{\text{obs}}) \mid \boldsymbol{X}_i\right] = \frac{\left(\mu_0(\boldsymbol{X}_i) - \tilde{\mu}_0(\boldsymbol{X}_i)\right)\left(\tilde{e}(\boldsymbol{X}_i) - e(\boldsymbol{X}_i)\right)}{1 - \tilde{e}(\boldsymbol{X}_i)} + \mu_0(\boldsymbol{X}_i)$$

  - If either the outcome or propensity score model is correct, we have

$$\mathbb{E}\left(f\left(w, \boldsymbol{X}_i, Y_i^{\text{obs}}\right)\right) = \mathbb{E}(Y_i(w) \mid \boldsymbol{X}_i)$$

- The DR estimator

$$\hat{\tau}^{\text{dr}} = \frac{1}{N}\sum_{i=1}^{N}\left\{\frac{W_i Y_i^{\text{obs}}}{\hat{e}(\boldsymbol{X}_i)} - \frac{W_i - \hat{e}(\boldsymbol{X}_i)}{\hat{e}(\boldsymbol{X}_i)}\hat{\mu}_1(\boldsymbol{X}_i)\right\} - \frac{1}{N}\sum_{i=1}^{N}\left\{\frac{(1 - W_i) Y_i^{\text{obs}}}{1 - \hat{e}(\boldsymbol{X}_i)} - \frac{\hat{e}(\boldsymbol{X}_i) - W_i}{1 - \hat{e}(\boldsymbol{X}_i)}\hat{\mu}_0(\boldsymbol{X}_i)\right\}$$

Comparison with other estimators

- The DR estimator

$$\hat{\tau}^{\mathrm{dr}} = \frac{1}{N}\sum_{i=1}^{N}\left\{\frac{W_i Y_i^{\mathrm{obs}}}{\hat{e}(\boldsymbol{X}_i)} - \frac{W_i - \hat{e}(\boldsymbol{X}_i)}{\hat{e}(\boldsymbol{X}_i)}\hat{\mu}_1(\boldsymbol{X}_i)\right\} - \frac{1}{N}\sum_{i=1}^{N}\left\{\frac{(1-W_i)Y_i^{\mathrm{obs}}}{1-\hat{e}(\boldsymbol{X}_i)} - \frac{\hat{e}(\boldsymbol{X}_i) - W_i}{1-\hat{e}(\boldsymbol{X}_i)}\hat{\mu}_0(\boldsymbol{X}_i)\right\}$$

- Comparison with the IPW estimator

$$\hat{\tau}^{\mathrm{dr}} = \hat{\tau}^{\mathrm{IPW}} - \frac{1}{N}\sum_{i=1}^{N}\frac{W_i - \hat{e}(\boldsymbol{X}_i)}{\hat{e}(\boldsymbol{X}_i)}\hat{\mu}_1(\boldsymbol{X}_i) + \frac{1}{N}\sum_{i=1}^{N}\frac{\hat{e}(\boldsymbol{X}_i) - W_i}{1-\hat{e}(\boldsymbol{X}_i)}\hat{\mu}_0(\boldsymbol{X}_i)$$

- Comparison with the outcome regression estimator

$$\hat{\tau}^{\mathrm{dr}} = \hat{\tau}^{\mathrm{reg}} + \frac{1}{N}\sum_{i=1}^{N}\frac{W_i}{\hat{e}(\boldsymbol{X}_i)}\left(Y_i^{\mathrm{obs}} - \hat{\mu}_1(\boldsymbol{X}_i)\right) - \frac{1}{N}\sum_{i=1}^{N}\frac{1-W_i}{1-\hat{e}(\boldsymbol{X}_i)}\left(Y_i^{\mathrm{obs}} - \hat{\mu}_0(\boldsymbol{X}_i)\right)$$

- Use bootstrap to compute the variance of $\hat{\tau}^{\mathrm{dr}}$

A simulation study (Kang and Schafer. 2007. Statistical Science)

- Setup:
  - 4 covariates Z_i: all are i.i.d. standard normal
  - Outcome model: linear model

$$y_i = 210 + 27.4 z_{i1} + 13.7 z_{i2} + 13.7 z_{i3} + 13.7 z_{i4} + \varepsilon_i$$

  - Propensity score model: logistic model with linear predictors

$$\pi_i = \text{expit}(-z_{i1} + 0.5 z_{i2} - 0.25 z_{i3} - 0.1 z_{i4})$$

  - Misspecification induced by measurement error:

$$x_{i1} = \exp(z_{i1}/2),$$

$$x_{i2} = z_{i2}/(1 + \exp(z_{i1})) + 10,$$

$$x_{i3} = (z_{i1} z_{i3}/25 + 0.6)^3,$$

$$x_{i4} = (z_2 + z_4 + 20)^2.$$

    - Corresponding outcome regression / propensity score model is mis-specified if Observe X_i instead of Z_i

A simulation study (Kang and Schafer. 2007. Statistical Science)

- The simulation reveals the deteriorating performance of propensity score weighting methods when the model is mis-specified

- Weighting estimators to be evaluated:
  - HT: IPW in the original form
  - IPW: IPW with normalized weights
  - WLS: Weighted least squares regression with covariates
    - IPW with normalization weights with some heuristic adjustment of covariates to improve efficiency
    - Not doubly robust

  - DR: Doubly-robust estimator

Results: if the propensity score model is correct

Sample size	Estimator	Bias logit	Bias True	RMSE logit	RMSE True
(1) Both models correct					
$n = 200$	HT	0.33	1.19	12.61	23.93
	IPW	−0.13	−0.13	3.98	5.03
	WLS	−0.04	−0.04	2.58	2.58
	DR	−0.04	−0.04	2.58	2.58
$n = 1000$	HT	0.01	−0.18	4.92	10.47
	IPW	0.01	−0.05	1.75	2.22
	WLS	0.01	0.01	1.14	1.14
	DR	0.01	0.01	1.14	1.14
(2) Propensity score model correct					
$n = 200$	HT	−0.05	−0.14	14.39	24.28
	IPW	−0.13	−0.18	4.08	4.97
	WLS	0.04	0.04	2.51	2.51
	DR	0.04	0.04	2.51	2.51
$n = 1000$	HT	−0.02	0.29	4.85	10.62
	IPW	0.02	−0.03	1.75	2.27
	WLS	0.04	0.04	1.14	1.14
	DR	0.04	0.04	1.14	1.14

- Normalizing weights can help a lot in reducing the variance
- WLS indeed gain efficiency even if outcome model is not linear
- Use the true propensity score is worse than using the estimated propensity score when the propensity score model is correct

Results: if the propensity score model is incorrect

		Bias		RMSE	
Sample size	Estimator	logit	True	logit	True
(3) Outcome model correct					
$n = 200$	HT	24.25	−0.18	194.58	23.24
	IPW	1.70	−0.26	9.75	4.93
	WLS	−2.29	0.41	4.03	3.31
	DR	−0.08	−0.10	2.67	2.58
$n = 1000$	HT	41.14	−0.23	238.14	10.42
	IPW	4.93	−0.02	11.44	2.21
	WLS	−2.94	0.20	3.29	1.47
	DR	0.02	0.01	1.89	1.13
(4) Both models incorrect					
$n = 200$	HT	30.32	−0.38	266.30	23.86
	IPW	1.93	−0.09	10.50	5.08
	WLS	−2.13	0.55	3.87	3.29
	DR	−7.46	0.37	50.30	3.74
$n = 1000$	HT	101.47	0.01	2371.18	10.53
	IPW	5.16	0.02	12.71	2.25
	WLS	−2.95	0.37	3.30	1.47
	DR	−48.66	0.08	1370.91	1.81

When only the outcome model is wrong

- Double robust estimator perform better when outcome model is correct but propensity score model is wrong
- WLS improves over IPW but not as good as DR

When both models are wrong

- Double robust estimator can perform worse when both models are wrong (maybe we should also normalize the weights in DR)