Lecture 6
Regression for completely randomized experiment

Outline

- Using regression with no covariates
- Using regression with covariates adjustments
- Using regression with covariates adjustments and interactions
- The LRC-CPPT cholesterol data example

Linear regression and causality

- Assume we have data from a completely randomized experiment
- We can run the following linear regression:

$$Y_i^{\text{obs}} \sim \alpha + \gamma W_i + \boldsymbol{\beta}^T \boldsymbol{X}_i$$

- Or a simpler linear regression model: $Y_i^{\text{obs}} \sim \alpha + \gamma W_i$
- Or more complicated linear regression model with interactions:

$$Y_i^{\text{obs}} \sim \alpha + \gamma W_i + \boldsymbol{\beta}^T \boldsymbol{X}_i + \boldsymbol{\gamma}^T \boldsymbol{X}_i W_i + \varepsilon_i$$

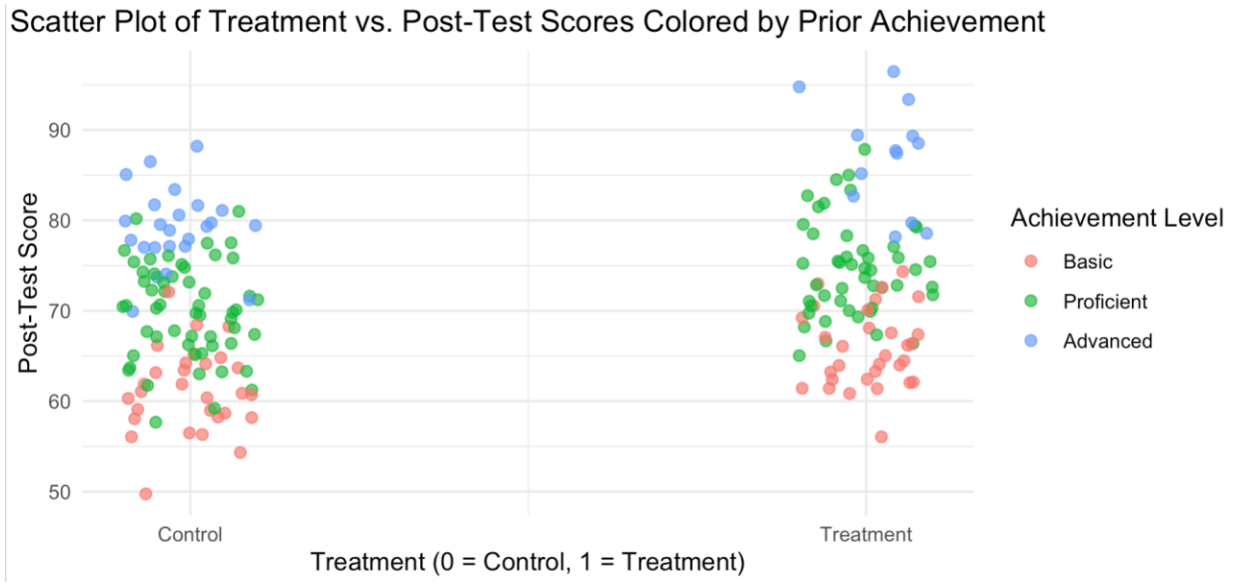
Or

$$Y_i^{\text{obs}} \sim \alpha + \gamma W_i + \boldsymbol{\beta}^T \boldsymbol{X}_i + \boldsymbol{\gamma}^T (\boldsymbol{X}_i - \bar{\boldsymbol{X}}) W_i + \varepsilon_i$$

- Which of these regression models provide an estimate of the average causal effect?
- Which model is the best to use?

Why linear model?

- Benefits of using linear regression:
  - Adjust for confounding variables
    - Not need for completely randomized experiments as pre-treatment covariates are not confounded
  - More accurate estimator if covariates explain part of the noise in the outcome



  - Widely used in practice
- How can we do correct inference if we take into account the randomization procedure of treatment assignments?

The LRC-CPPT cholesterol data

- An experiment to evaluate the effect of the drug cholestyramine on reducing cholesterol levels
- $N = 337$ patients are completely randomized
- **Pre-treatment covariates:** two cholesterol measurements before and after a suggestion of low-cholesterol diet, both measurements taken prior to the random assignment
  - cholp = 0.25 chol1 + 0.75 chol2

Table 7.1. ***Summary Statistics for PRC-CPPT Cholesterol Data***

	Variable	Control (N_c =172)		Treatment (N_t =165)			
		Average	Sample (S.D.)	Average	Sample (S.D.)	Min	Max
Pre-treatment	`chol1`	297.1	(23.1)	297.0	(20.4)	247.0	442.0
	`chol2`	289.2	(24.1)	287.4	(21.4)	224.0	435.0
	`cholp`	291.2	(23.2)	289.9	(20.4)	233.0	436.8
Post-treatment	`cholf`	282.7	(24.9)	256.5	(26.2)	167.0	427.0
	`chold`	−8.5	(10.8)	−33.4	(21.3)	−113.3	29.5
	`comp`	74.5	(21.0)	59.9	(24.4)	0	101.0

The LRC-CPPT cholesterol data

- An experiment to evaluate the effect of the drug cholestyramine on reducing cholesterol levels
- $N = 337$ patients are completely randomized
- **Post-treatment outcomes:**
  - cholf: post-treatment average cholesterol level
  - chold = cholf – cholp
  - comp: compliance rate, the percentage of individuals follow the treatment assignment

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The LRC-CPPT cholesterol data

- Can we evaluate the drug effect by simply look at whether chold is positive or negative?
  - No! The before-after comparison is NOT necessarily causal
  - Even for the control group, chold is significantly negative
- The patient's post-treatment cholesterol should be highly correlated with his/her pre-treatment cholesterol level
- How do we evaluate the causal effect after "adjusting for the pre-treatment cholesterol"?
  - Adjust for pre-treatment cholesterol by regression

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Linear regression with no covariates

- Model:

$$Y_i^{\text{obs}} = a + bW_i + \varepsilon_i$$

- What is the solution?

$$(\hat{a}, \hat{b}) = \arg\min_{(a,b)} \sum_{i=1}^{N} (Y_i^{\text{obs}} - a - bW_i)^2$$

  - $\arg\min_{(a,b)} \sum_{i=1}^{n} (Y_i^{\text{obs}} - a - bW_i)^2 = \arg\min_{(a,b)} \left[\sum_{i:W_i=0} (Y_i^{\text{obs}} - a)^2 + \sum_{i:W_i=1} (Y_i^{\text{obs}} - a - b)^2\right]$
  - $\overline{Y}_{\text{c}}^{\text{obs}} = \frac{1}{N_{\text{c}}} \sum_{i:W_i=0} Y_i^{\text{obs}}$ and $\overline{Y}_{\text{t}}^{\text{obs}} = \frac{1}{N_{\text{t}}} \sum_{i:W_i=1} Y_i^{\text{obs}}$
  - $\hat{a} = \bar{Y}_c^{\text{obs}}$, $\hat{a} + \hat{b} = \bar{Y}_t^{\text{obs}}$
  - $\hat{b} = \bar{Y}_t^{\text{obs}} - \bar{Y}_c^{\text{obs}}$

- Same as the estimator from Neyman's approach $\hat{\tau}^{\text{dif}} = \overline{Y}_{\text{t}}^{\text{obs}} - \overline{Y}_{\text{c}}^{\text{obs}}$

Causal interpretation of this linear model

- Assume that there is a super-population and the potential outcomes $(Y_i(0), Y_i(1))$ are i.i.d. samples

- Linear model on the potential outcomes

$$Y_i(0) = \alpha + \varepsilon_i(0)$$

$$Y_i(1) = Y_i(0) + \tau_i = \alpha + \tau + \varepsilon_i(0) + (\tau_i - \tau) = \alpha + \tau + \varepsilon_i(1)$$

  - $\tau = \mathbb{E}(\tau_i)$, $\mathbb{E}\big(\varepsilon_i(0)\big) = 0$

- In general, we have the model

$$Y_i(w) = \alpha + \tau w + \varepsilon_i(w)$$

  - $\alpha = \mathbb{E}\big(Y_i(0)\big)$ and $\mathbb{E}\big(\varepsilon_i(w)\big) = 0$

- If the treatment is binary ($w = 0,1$), then the above model essentially has no assumption on $Y_i(0)$ and $Y_i(1)$
- If the treatment is continuous, the model assumes a linear but heterogenous causal effect on each individual

Causal interpretation of this linear model

- The observed outcomes $Y_i = W_i Y_i(1) + (1 - W_i) Y_i(0)$
  - Are observed Y_i i.i.d. samples under complete randomization?
  - Conditional on $(W_1 \cdots, W_n)$, are Y_i i.i.d. , are Y_i independent across i?

We assume the following identification conditions

- Randomization of the treatment (unconfoundedness):

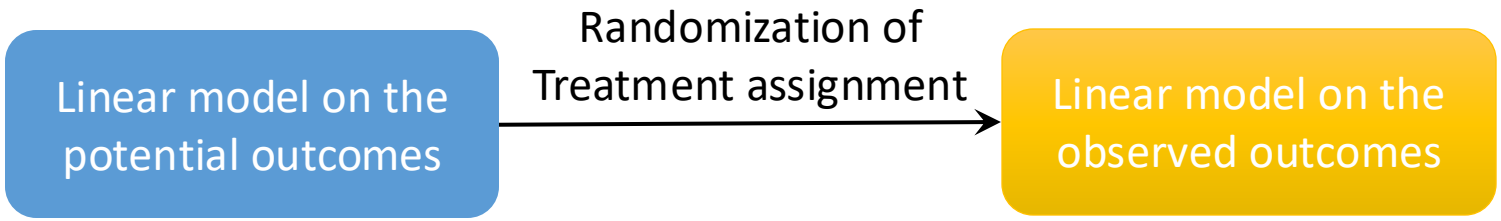
$$(\boldsymbol{Y}(0), \boldsymbol{Y}(1)) \perp \boldsymbol{W}$$

  - Satisfied in completely randomized experiments
  - Then, $\mathbb{E}\left(Y_i^{\text{obs}} | W_i = w\right) = \mathbb{E}(Y_i(w) | W_i = w) = \mathbb{E}\left(Y_i(w)\right) = \alpha + \tau w$

- A linear regression model on observed outcome:

$$Y_i^{\text{obs}} = \alpha + \tau W_i + \varepsilon_i$$

  where $\varepsilon_i = \varepsilon_i(W_i)$



Linear regression with no covariates

- regression model

$$Y_i^{\text{obs}} = \alpha + \tau W_i + \varepsilon_i$$

  where $\varepsilon_i = \varepsilon_i(W_i)$

How to perform statistical inference?

- Follow the linear regression convention, we perform statistical inference conditional on $(W_1, \cdots, W_N)$
  - we treat assignment vectors as fixed

- Random sampling of the units
  - $(\varepsilon_i(0), \varepsilon_i(1))$ are independent across i
  - This implies that ε_i in the linear regression model are independent as W_i are treated as fixed
  - But they may not follow the same distribution

Homoscedastic error assumption

Homoscedastic error assumption: $\mathbb{V}(\varepsilon_i(0)) = \mathbb{V}(\varepsilon_i(1)) = \sigma^2$

- Then $\mathbb{V}(Y_i^{\text{obs}}|W_i) = \varepsilon_i = \varepsilon_i(W_i)$ always has variance σ^2
- Under homoscedasticity, OLS estimates of the variance is

$$\hat{\sigma}^2_{Y|W} = \frac{1}{N-2}\sum_{i=1}^{N}\hat{\varepsilon}_i^2 = \frac{1}{N-2}\sum_{i=1}^{N}\left(Y_i^{\text{obs}} - \hat{Y}_i^{\text{obs}}\right)^2,$$

where the estimated residual is $\hat{\varepsilon}_i = Y_i^{\text{obs}} - \hat{Y}_i^{\text{obs}}$, and the predicted value $\hat{Y}_i^{\text{obs}}$ is

$$\hat{Y}_i^{\text{obs}} = \begin{cases} \hat{\alpha}^{\text{ols}} & \text{if } W_i = 0, \\ \hat{\alpha}^{\text{ols}} + \hat{\tau}^{\text{ols}} & \text{if } W_i = 1. \end{cases}$$

  - Same as the standard linear regression approach
    - variance estimator for $\hat{\tau}$ is $\frac{1}{N}\frac{(N_c-1)s_c^2+(N_t-1)s_t^2}{N-2}$

Heteroscedastic errors

- If we don't want to assume $\mathbb{V}(\varepsilon_i(0)) = \mathbb{V}(\varepsilon_i(1))$, then the homoscedastic error assumption fails
  - ε_i has the same distribution for $W_i = 0$, and the same distribution for $W_i = 1$
  - We should use same variance within the treated and control group
  - That leads to the variance estimator of $\hat{\tau}$ as $\frac{s_c^2}{N_c} + \frac{s_t^2}{N_t}$
    - Same as Neyman's approach
- This is also called the Sandwich estimator that is robust to the violation of the homoscedastic noise assumption in linear regression
  - In R, it corresponds to Sandwich estimator with HC2 adjustment

Linear regression wit no covariates

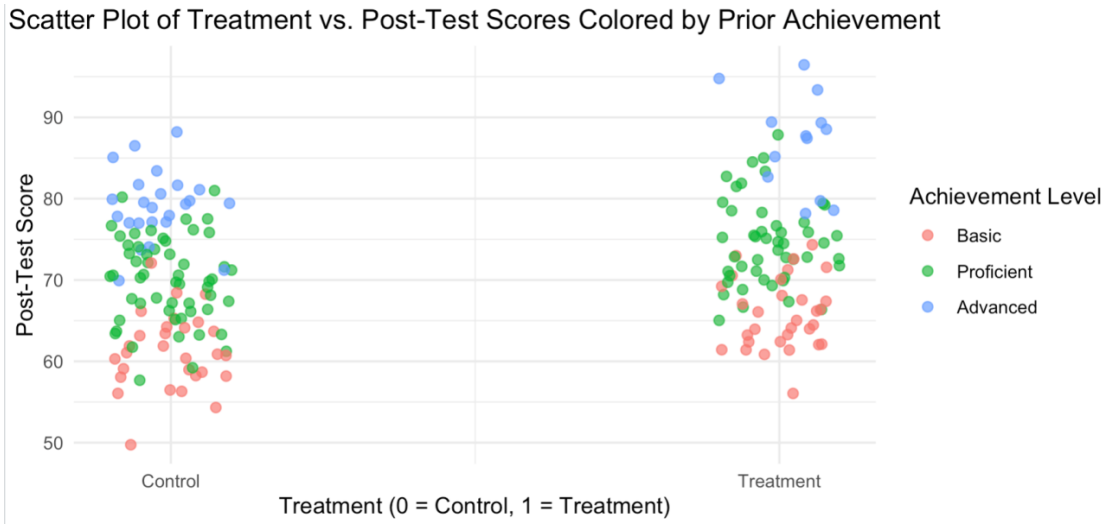
To summarize the logic

- We build a (linear) model on the potential outcomes
- This model implies a linear regression model on the observed outcome if $(\boldsymbol{Y}(0), \boldsymbol{Y}(1)) \perp \boldsymbol{W}$
- The coefficient on W_i in the linear regression model is the average causal effect (PATE)
- The linear regression model treat $\boldsymbol{W}$ as fixed so it works for any randomization assignment mechanism that satisfy $(\boldsymbol{Y}(0), \boldsymbol{Y}(1)) \perp \boldsymbol{W}$
- Noise in the linear regression model are independent as long as potential outcomes are independent across units

- For statistical inference
  - The OLS estimator estimator is always unbiased
  - We can apply standard linear regression inference results if we assume $\mathbb{V}(\varepsilon_i(0)) = \mathbb{V}(\varepsilon_i(1))$
  - If $\mathbb{V}(\varepsilon_i(0)) \neq \mathbb{V}(\varepsilon_i(1))$, we need to use the robust variance estimator

Linear regression with covariates adjustment

- $Y_i^{\text{obs}} = \alpha + \tau W_i + \boldsymbol{\beta}^T \boldsymbol{X}_i + \varepsilon_i$
  - Why may we prefer adjusting for $\boldsymbol{X}_i$?



- What is the corresponding model on potential outcomes?
- Do they always increase efficiency?

Causal interpretation of this linear model

- Assumption 1: $\mathbb{E}(Y_i(0)|\, \boldsymbol{X}_i) = \alpha + \boldsymbol{\beta}^T \boldsymbol{X}_i$
- Assumption 2: CATE $\tau(\boldsymbol{x}) = \mathbb{E}(\tau_i|\, \boldsymbol{X}_i = \boldsymbol{x}) \equiv \tau =$ PATE constant across levels of $\boldsymbol{X}_i$
  - We can allow for heterogeneous causal effect but need $\mathbb{E}(\tau_i - \tau \mid \boldsymbol{X}_i) = 0$ (individual causal effects are independent from the pre-treatment covariates)

- Then $\mathbb{E}(Y_i(w)|\, \boldsymbol{X}_i) = \mathbb{E}(Y_i(0) + \tau_i w|\, \boldsymbol{X}_i) = \alpha + \tau w + \boldsymbol{\beta}^T \boldsymbol{X}_i$

- Under unconfoundedness property: $\big(\boldsymbol{Y}(0), \boldsymbol{Y}(1)\big) \perp \boldsymbol{W} \mid \boldsymbol{X}$
  - $\mathbb{E}\big(Y_i^{\text{obs}}|W_i = w, \boldsymbol{X}_i = \boldsymbol{x}\big) = \mathbb{E}(Y_i(w)|\boldsymbol{X}_i = \boldsymbol{x}) = \alpha + \tau w + \boldsymbol{\beta}^T \boldsymbol{X}_i$
    - Statistical inference is conditional on both $\boldsymbol{X}_i$ and W_i

  - Even if the causal model is incorrect (either the violation of $\mathbb{E}(Y_i(0)|\, \boldsymbol{X}_i) = \alpha + \boldsymbol{\beta}^T \boldsymbol{X}_i$ or $\tau \equiv \mathbb{E}(\tau_i|\, \boldsymbol{X}_i = \boldsymbol{x})$), this regression still gives valid estimation of τ under complete randomization (see next page)

OLS with covariates adjustment

$$(\hat{\alpha}^{\text{ols}}, \hat{\tau}^{\text{ols}}, \hat{\beta}^{\text{ols}}) = \arg\min_{\alpha,\tau,\beta} \sum_{i=1}^{N} \left(Y_i^{\text{obs}} - \alpha - \tau \cdot W_i - X_i\beta \right)^2$$

- The estimator $\hat{\tau}^{\text{ols}}$ is unbiased for the causal estimand τ
- Even if the model is incorrect, $\hat{\tau}^{\text{ols}}$ still converges to τ under complete randomization
  - $\hat{\tau}^{\text{ols}} = \bar{Y}_t^{\text{obs}} - \bar{Y}_c^{\text{obs}} - \widehat{\boldsymbol{\beta}}^T(\bar{\boldsymbol{X}}_t - \bar{\boldsymbol{X}}_c)$

Efficiency gain from regression

- If the model is correct, we have

$$\mathbb{V}\left(\hat{\tau}^{\text{ols}}\right) \approx \frac{\mathbb{E}\{\mathbb{V}(Y_i(1)|\,\boldsymbol{X}_i)\}}{N_t} + \frac{\mathbb{E}\{\mathbb{V}(Y_i(0)|\,\boldsymbol{X}_i)\}}{N_c} \leq \frac{\sigma_c^2}{N_c} + \frac{\sigma_t^2}{N_t}$$

  - If $\boldsymbol{X}_i$ is predictive of the (potential) outcomes, we have a more accurate estimator
- If the linear model is incorrect, the efficiency might be lost (Freedman 2008, *Adv. Appl. Math.)*

Estimate of the variance of $\hat{\tau}^{\text{ols}}$ with covariates adjustment

- Assume homoscedastic error assumption:

$$\mathbb{V}(\varepsilon_i(0)) = \mathbb{V}(\varepsilon_i(1)) = \sigma^2 = \mathbb{V}\left(Y_i^{\text{obs}} | W_i, \boldsymbol{X}_i\right)$$

  We can follow standard linear regression inference and estimate variance of $\hat{\tau}^{\text{ols}}$ as

$$\hat{\mathbb{V}}_{\text{sp}}^{\text{homo}} = \frac{1}{N\,(N-1-\dim(X_i))} \cdot \frac{\sum_{i=1}^{N}\left(Y_i^{\text{obs}} - \hat{\alpha}^{\text{ols}} - \hat{\tau}^{\text{ols}} - X_i\hat{\beta}^{\text{ols}}\right)^2}{\overline{W}\cdot(1-\overline{W})}$$

- The robust variance estimator (Sandwich estimator) without assuming homoscedasticity

$$\hat{\mathbb{V}}_{\text{sp}}^{\text{hetero}} = \frac{1}{N\,(N-1-\dim(X_i))} \cdot \frac{\sum_{i=1}^{N}\left(W_i-\overline{W}\right)^2\cdot\left(Y_i^{\text{obs}} - \hat{\alpha}^{\text{ols}} - \hat{\tau}^{\text{ols}} - X_i\hat{\beta}^{\text{ols}}\right)^2}{\left(\overline{W}\cdot(1-\overline{W})\right)^2}$$

Linear regression with covariates adjustment and interactions

Is this assumption $\tau \equiv \tau(\boldsymbol{x}) = \mathbb{E}(\tau_i | \ \boldsymbol{X}_i = \boldsymbol{x})$ reasonable?

- Effect heterogeneity across gender, age, pre-existing conditions …
- How do we allow such heterogeneity in linear regression?

- Assume CATE $\tau(\boldsymbol{x}) = \mathbb{E}(\tau_i | \ \boldsymbol{X}_i = \boldsymbol{x}) = \tau + \boldsymbol{\gamma}^T(\boldsymbol{x} - \bar{\boldsymbol{X}})$
  - τ is still the population average treatment effect
  - Why do we need centering?
    - If we assume $\mathbb{E}(\tau_i | \ \boldsymbol{X}_i = \boldsymbol{x}) = \tau + \boldsymbol{\gamma}^T \boldsymbol{x}$, then $\mathbb{E}(\tau_i) = \tau + \boldsymbol{\gamma}^T \mathbb{E}(\boldsymbol{X}_i)$

- Still assume $\mathbb{E}(Y_i(0) | \ \boldsymbol{X}_i) = \alpha + \boldsymbol{\beta}^T \boldsymbol{X}_i$
- Then

$$\mathbb{E}(Y_i(w) | \ \boldsymbol{X}_i) = \mathbb{E}(Y_i(0) + \tau_i w | \ \boldsymbol{X}_i) = \alpha + \tau w + \boldsymbol{\beta}^T \boldsymbol{X}_i + \boldsymbol{\gamma}^T (\boldsymbol{X}_i - \bar{\boldsymbol{X}}) w$$

Linear regression with covariates adjustment and interactions

- When does the above model imply the same model on observed data?
  - Under unconfoundedness: $\left(\boldsymbol{Y}(0), \boldsymbol{Y}(1)\right) \perp \boldsymbol{W} \mid \boldsymbol{X}$

    $\mathbb{E}\left(Y_i^{\text{obs}} | W_i = w, \boldsymbol{X}_i = \boldsymbol{x}\right) = \mathbb{E}(Y_i(w) | \boldsymbol{X}_i = \boldsymbol{x}) = \alpha + \tau w + \boldsymbol{\beta}^T \boldsymbol{X}_i + \boldsymbol{\gamma}^T (\boldsymbol{X}_i - \bar{\boldsymbol{X}}) w$
    - Statistical inference is conditional on both $\boldsymbol{X}_i$ and W_i

$$Y_i^{\text{obs}} = \alpha + \tau W_i + \boldsymbol{\beta}^T \boldsymbol{X}_i + \boldsymbol{\gamma}^T (\boldsymbol{X}_i - \bar{\boldsymbol{X}}) W_i + \varepsilon_i$$

- What is the benefit of adding interactions
  - More flexible model assumptions
  - Further increase efficiency
  - In completely randomized experiments, with the interaction terms, we can always guarantee no efficiency loss even when the linear model is wrong (Peng's book section 6.2.2)

Results on the LRC-CPPT cholesterol data

- We estimate the PATE for both the post-treatment cholesterol level cholf and compliance
  - A considerable reduction of the variance of $\hat{\tau}^{\text{ols}}$ for cholf when we add the pre-treatment cholesterol levels in the regression
  - Our goal is always estimating PATE even after "covariates adjustment"
  - In randomized experiments satisfying $(\boldsymbol{Y}(0), \boldsymbol{Y}(1)) \perp \boldsymbol{W}$, adjusting for covariates or not, our estimate of PATE is always valid, we only change the efficiency of our estimate

Covariates	Effect of Assignment to Treatment on Post-Cholesterol Level		Compliance	
	$\hat{\tau}$	$\widehat{(\text{s. e.})}$	$\hat{\tau}$	$\widehat{(\text{s. e.})}$
No covariates	−26.22	(3.93)	−14.64	(3.51)
`cholp`	−25.01	(2.60)	−14.68	(3.51)
`chol1, chol2`	−25.02	(2.59)	−14.95	(3.50)
`chol1, chol2`, interacted with W	−25.04	(2.56)	−14.94	(3.49)

The LRC-CPPT cholesterol data

A bit explanation about compliance

- If we compare between control and treatment group, we are evaluating the causal effect of "being assigned", not the causal effect of actually taking the drug
- Compliance lower in the treatment group possibly due to the side effect of the drug
- Can we just throw away individuals who do not follow the treatment and estimate the causal effect of taking the drug based on the rest individuals? No
- Will discuss more about compliance in later lectures

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Why do we use linear regression in randomized experiments?

- Covariate adjustment can be used to improve efficiency in randomized experiments
  - Always add interaction terms (between each covariate and treatment) to guarantee power improvement

- In completely randomized experiments
  - No need to worry about model misspecification
  - Treatment and covariates are independent