# STAT347: Generalized Linear Models
# Lecture 6

Today's topics: Chapters 5.3 - 5.5, 5.7

- Binary GLM inference
- Fitting logistic regression and the infinite estimates
- Binary GLM example

## 1 Binary GLM model inference

We have already learnt the inference of a general GLM model, we now look what the specific forms are for a binary GLM.

### 1.1 Score equation in logistic regression

For logistic regression, as the logit link is the canonical link, the score equation is:

$$\frac{\partial L}{\partial \beta_j} = \sum_i (y_i - n_i p_i) x_{ij} = \sum_i \left( y_i - \frac{n_i e^{X_i^T \beta}}{1 + e^{X_i^T \beta}} \right) x_{ij} = 0$$

We have derived that as $n \to \infty$

$$\text{Var}(\hat{\beta}) \to (X^T W X)^{-1}$$

where $W = D^2 V^{-1}$ is a diagonal matrix. For logistic regression where the logit link is the canonical link, we have $W = V$ so

$$W_{ii} = n_i p_i (1 - p_i), \quad \widehat{W}_{ii} = n_i \frac{e^{X_i^T \hat{\beta}}}{(1 + e^{X_i^T \hat{\beta}})^2}$$

### 1.2 Hypothesis testing

Consider the simple null model for binomial data we discussed earlier. Under the null model, the group data is $\sum_i y_i \sim \text{Binomial}(N, p)$ which has only one sample. We want to test for $H_0 : \beta = \text{logit}(p_0)$ (or equivalently: $H_0 : p \equiv p_0$) where $\beta$ is the constant coefficient. Define $y = \sum_i y_i / N$, under the null model, we can quickly find the MLE, which is $\hat{p} = y$ and $\hat{\beta} = \text{logit}(y)$.

The test statistics are

Wald test:

$$\left( \frac{\hat{\beta} - \text{logit}(p_0)}{\widehat{\text{SE}}(\hat{\beta})} \right)^2 = [\text{logit}(y) - \text{logit}(p_0)]^2 N y (1 - y),$$

Or

$$\left(\frac{\hat{p}-p_0}{\widehat{\mathrm{SE}}(\hat{p})}\right)^2 = \frac{(y-p_0)^2}{[y(1-y)/N]}$$

Likelihood ratio test:

$$-2(L_0 - L_1) = -2\log\left[\frac{p_0^{Ny}(1-p_0)^{N-Ny}}{y^{Ny}(1-y)^{N-Ny}}\right]$$

Score test:

$$T = \frac{\dot{L}(\beta_0)^T \dot{L}(\beta_0)}{-\ddot{L}(\beta_0)} = \frac{(y-p_0)^2}{[p_0(1-p_0)/N]}$$

- Wald test depends on the scale
- Wald test is less stable when $y$ is close to 0 or 1. Read Chapter 5.3.3

### 1.3 Deviance

The total (residual) deviance for a binary GLM (the deviance between the saturated model and the fitted model) is

$$\begin{aligned}
D_+(y,\hat{\mu}) &= \sum_i D(y_i, n_i\hat{p}_i) \\
&= -2\sum_i \log\left[f(y_i,\hat{\theta}_i)/f(y_i,\theta_{y_i})\right] \\
&= -2\sum_i \log\left[\frac{\hat{p}_i^{y_i}(1-\hat{p}_i)^{n_i-y_i}}{(y_i/n_i)^{y_i}(1-y_i/n_i)^{n_i-y_i}}\right] \\
&= 2\sum_i y_i \log\frac{y_i}{n_i\hat{p}_i} + 2\sum_i (n_i-y_i)\log\frac{n_i-y_i}{n_i-n_i\hat{p}_i}
\end{aligned}$$

- The total deviance is different for grouped data and ungrouped data as the saturated model is different.
  - Ungrouped data: the saturated model is $\hat{p}_i = y_i$ for each individual sample
  - grouped data: the saturated model is $\hat{p}_k = \tilde{y}_k$ for each group $k$. Thus all samples in the same group should have the same $\hat{p}_i$ even in the saturated model.

## 2 Binary GLM computation

For logistic regression, Newton's method = Fisher scoring = IRLS.

For IRLS, the $t$th iteration is

$$X^T W^{(t)}(z^{(t)} - X\beta) = 0$$

where

$$\begin{aligned}
z_i^{(t)} &= X_i^T\beta^{(t)} + \left(D_{ii}^{(t)}\right)^{-1}(y_i - \mu_i^{(t)}) \\
&= \log\left(\frac{p_i^{(t)}}{1-p_i^{(t)}}\right) + \frac{y_i - n_i p_i^{(t)}}{n_i p_i^{(t)}(1-p_i^{(t)})}
\end{aligned}$$

and

$$W_{ii}^{(t)} = V_{ii}^{(t)} = n_i p_i^{(t)}(1-p_i^{(t)})$$

### 2.1 Infinite parameter estimates

One may sometimes see this warning message using R to solve the logistic regression:

*Warning message: glm.fit: fitted probabilities numerically 0 or 1 occurred*

You may see very large estimates of $\beta$. What happened?

- Perfect separation:

  There exists $\beta_s$ such that if $X_i^T\beta_s > 0$ then $y_i = 1$ otherwise $y_i = 0$.

  We proof that the MLE for $\beta$ does not exist. Let $\eta_i = kX_i^T\beta_s$.

  When $k \to \infty$, then

  $$p_i = \frac{e^{kX_i^T\beta_s}}{1 + e^{kX_i^T\beta_s}} \to \begin{cases} 1 & \text{if } X_i^T\beta_s > 0, \text{ or equivalently } y_i = 1 \\ 0 & \text{else} \end{cases}$$

  Thus, $\frac{\partial L}{\partial \beta} \to 0$ if $k \to \infty$ so the solution of the score equation is infinite. In other words, the MLE does not exist.

- Quasi-complete separation:

  There exists $\beta_s$ such that if $X_i^T\beta_s > 0$ then $y_i = 1$, if $X_i^T\beta_s < 0$ then $y_i = 0$, and if $X_i^T\beta_s = 0$ then $y_i = 0$ or 1 (allow data points on the separation hyperplane with both outcomes).

  We can also show that the MLE for $\beta$ does not exist (Albert and Anderson, *Biometrika* 1984). Any value $\beta$ can be decomposed as $\beta = \beta_s + \gamma$. Denote $\beta_k = k\beta_s + \gamma$ Let $\eta_i = kX_i^T\beta_s + X_i^T\gamma$. When $k \to \infty$, then

  $$p_i = \frac{e^{kX_i^T\beta_s + X_i^T\gamma}}{1 + e^{kX_i^T\beta_s + X_i^T\gamma}} \to \begin{cases} 1 & \text{if } X_i^T\beta_s > 0 \\ 0 & \text{if } X_i^T\beta_s < 0 \\ \frac{e^{X_i^T\gamma}}{1+e^{X_i^T\gamma}} & \text{if } X_i^T\beta_s = 0 \end{cases}$$

  This tells us that for any $\beta$, we can find $\beta_k$ with large enough $k$ so that the log-likelihood $L(\beta_k) > L(\beta)$, so the log-likelihood function $L(\cdot)$ does not have a finite maximum point. In other words, the MLE does not exist.

- How to deal with perfect/quasi-complete separation? (Read Chapter 5.4.2)

  We can add a penalization or add a prior of the parameter to obtain finite estimates of $\beta$.

## 3 Two data examples

Chapter 5.7. Please check the R notebook 3.

Next time: Chapter 6.1, multivariate GLM: nominal response