

STAT347: Generalized Linear Models

Lecture 12

Today's topics: Chapters 8

- Quasi-likelihood
- Estimating equations and the Sandwich estimator

1 Quasi-likelihood

The above solution replaces the exponential family distributions with a more complicated parametric distribution allowing an extra dispersion parameter in the variance. Another more general solution is to only assume a mean-variance relationship.

Remind the the score equation for the exponential family distributed data is:

$$\frac{\partial L}{\partial \beta_j} = \sum_i \frac{(y_i - \mu_i)x_{ij}}{\text{Var}(y_i)} \frac{1}{g'(\mu_i)} = 0$$

- These score equations only involve $E(y_i) = \mu_i$ and $\text{Var}(y_i)$.
- Quasi-likelihood: we replace $\text{Var}(y_i)$ by some other mean-variance relationship that we believe can better fit the data.
- Typically, the mean-variance relationship can involves another unknown dispersion parameter.
- Here, we DO NOT assume any other aspects of the distribution of y_i besides mean and variance.

Common forms of mean-variance relationship $\text{Var}(y_i) = a(\mu_i, \phi)$:

- Proportional: $a(\mu_i, \phi) = \phi v^*(\mu_i)$.
 - counts: assume $a(\mu_i, \phi) = \phi \mu_i$
 - grouped Binary data: $a(\mu_i, \phi) = \phi \mu_i (n_i - \mu_i) / n_i$
- For counts we can also assume $a(\mu_i, \phi) = \mu_i + \phi \mu_i^2$ as in the Negative-Binomial distribution
- For grouped Binary data we can also assume $a(\mu_i, \phi) = \mu_i (n_i - \mu_i) (1 + (n_i - 1)\phi)$ as in the Beta-Binomial distribution

Some related properties:

- The proportional mean-variance relationship is the easiest for the computation of $\hat{\beta}$ as ϕ cancels and does not affect solving the score equations for β .
- $\text{Var}(\hat{\beta})$ is affected by ϕ for any of the above mean-variance relationships.

- Including ϕ helps to get a correct uncertainty quantification of $\hat{\beta}$.

How to estimate ϕ ? As we don't know the likelihood of the data, we only use moment conditions.

- When $a(\mu_i, \phi) = \phi v^*(\mu_i)$, we can get $\hat{\beta}$ thus $\hat{\mu}_i$ first without knowing ϕ . Then define

$$X^2 = \sum_{i=1}^n \frac{(y_i - \hat{\mu}_i)^2}{\phi v^*(\hat{\mu}_i)}$$

We can solve ϕ by solving $X^2 = n - p$ (we use $n - p$ instead of n to correct for the degree of freedom in the estimated $\hat{\mu}_i$), which is

$$\hat{\phi} = \frac{1}{n - p} \sum_{i=1}^n \frac{(y_i - \hat{\mu}_i)^2}{v^*(\hat{\mu}_i)}$$

- For other forms of $a(\mu, \phi)$, we need to solve ϕ and β simultaneously from equations

$$\varphi_{1j}(\beta, \phi) = \frac{\partial L}{\partial \beta_j} = \sum_i \frac{(y_i - \mu_i) x_{ij}}{a(\mu_i, \phi)} \frac{1}{g'(\mu_i)} = 0 \quad (1)$$

$$\varphi_2(\beta, \phi) = \sum_{i=1}^n \frac{(y_i - \mu_i)^2}{a(\mu_i, \phi)} - (n - p) = 0 \quad (2)$$

- $\mathbb{E}[\varphi_{1j}(\beta, \phi)] = 0$ and $\mathbb{E}[\varphi_2(\beta, \phi)]/n \rightarrow 0$. Solutions $\hat{\beta}$ and $\hat{\phi}$ are called Z-estimators. Under proper regularity conditions, we can show that both $\hat{\beta}$ and $\hat{\phi}$ are consistent.

2 Estimating equations and Sandwich estimator

How to estimate the variance of $\hat{\beta}$ from the quasi-likelihood equations? And what if we do not even know the true form of the mean-variance relationship?

- The equations (2) is one type of estimating equations. In general, the estimating equations for parameters θ (here $\theta = (\beta, \phi)$ or $\theta = \beta$) have the form:

$$u(\theta) = \sum_i u_i(\theta) = 0$$

Denote the solution of these equations as $\hat{\theta}$ and the true θ as θ_0 .

- Consistency: roughly speaking, when p is small, if $E(u(\theta_0)) \rightarrow 0$ when $n \rightarrow \infty$, then we can have $\hat{\theta} \rightarrow \theta_0$ (with some additional conditions).
- Variance of $\hat{\theta}$. Under consistency, we can estimate the asymptotic variance of $\hat{\theta}$ by first-order Taylor expansion (see later).
- The score equations

$$u(\beta) = \sum_i \frac{(y_i - \mu_i) x_{ij}}{v^*(\mu_i)} \frac{1}{g'(\mu_i)} = 0$$

are valid estimating equations ($\mathbb{E}[u(\beta_0)] = 0$) as long as the link function is correct. The response y_i does not need to follow the assumed exponential family distribution and $v^*(\mu_i)$ does not need to be the correct form of variance.

- Even the simple $\sum_i (y_i - \mu_i) x_{ij} = 0$ are always valid estimating equations. The problem is that $\text{sd}(\hat{\beta})$ may be large if samples have unequal variances.

Sandwich estimator of the asymptotic variances:

Let's now calculate the asymptotic variance of $\hat{\theta}$ for

$$\mu(\hat{\theta}) = 0$$

By first-order Taylor expansion, we have

$$0 = u(\hat{\theta}) \approx u(\theta_0) + \dot{u}(\theta_0)(\hat{\theta} - \theta_0)$$

Thus, we have

$$\hat{\theta} - \theta_0 \approx -\dot{u}(\theta_0)^{-1} u(\theta_0)$$

Roughly speaking, we have

- Law of large numbers:

$$\frac{1}{n} \dot{u}(\theta_0) = \frac{1}{n} \sum_{i=1}^n \dot{u}_i(\theta_0) \rightarrow E \left(\frac{1}{n} \sum_{i=1}^n \dot{u}_i(\theta_0) \right) = A$$

- CLT:

$$\frac{1}{\sqrt{n}} u(\theta_0) = \frac{1}{\sqrt{n}} \sum_{i=1}^n u_i(\theta_0) \approx N(0, V)$$

Thus

$$\text{Var}(\hat{\theta}) \approx A^{-1} V A^{-T} / n$$

In practice, we can estimate A and V by

$$\hat{A} = \frac{1}{n} \sum_{i=1}^n \dot{u}_i(\hat{\theta})$$

and

$$\hat{V} = \frac{1}{n} \sum_i u_i(\hat{\theta}) u_i(\hat{\theta})^T$$

- We use the sample variance to approximate V without knowing the distribution of the data
- The Sandwich estimator provides an estimate of the variance of $\hat{\beta}$ even when model assumption is violated.

3 Revisit the horseshoe crab data

Please see R notebook Example 7.

Next time: Mixed effect linear models