Topics in Causal Inference

STAT41530

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Lecture 9

Topic:
Lord paradox

- What is Lord paradox?
- Three interpretations
  - An interpretation without using causal language [Cox and McCullagh, 1982]
  - Potential outcome interpretation [Holland and Rubin, 1983]
  - Interpretation as mediation analysis [Pearl, 2016]

Original statement of Lord paradox

[Lord 1967, 1969, 1975]

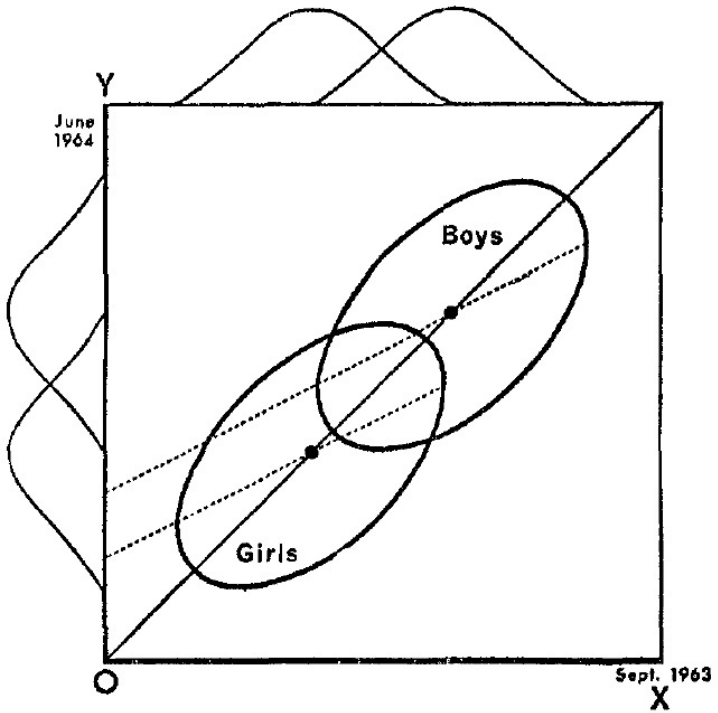


FIG. 1. Hypothetical scatterplots showing initial and final weight for boys and for girls.

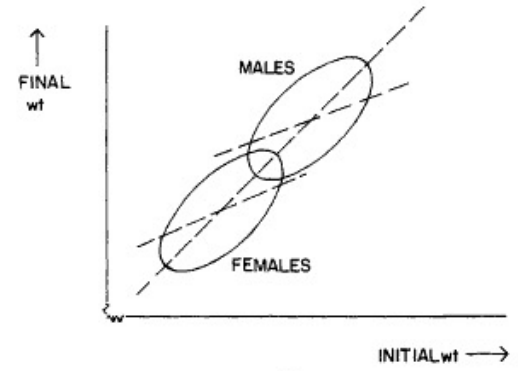
Ellipses represent separate scatter plots

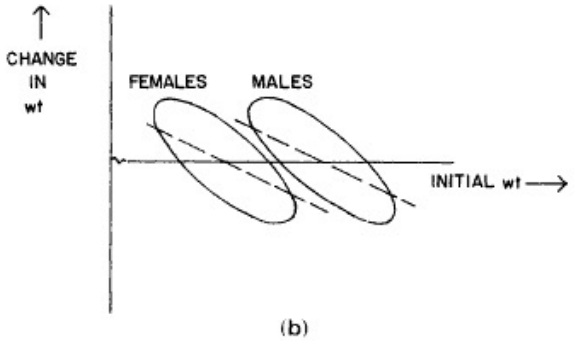
- A large university is interested in investigating the effects on the students of the diet provided in the university dining halls and any sex difference in these effects.
- The weight of each student at the time of his arrival in September (X) and his weight the following June (Y) are recorded.

- First statistician
  - Distribution of weight of the girls at the beginning of the year and at the end of the year are identical. Same for the boys.
  - no evidence of any interesting effect of the school diet on student weight. No evidence of any differential effect on the two sexes

- Second statistician
  - Linear regression of Y on both X and gender and coefficient of gender > 0
  - If initial weight a subgroup of boys and a subgroup of girls having identical, then the subgroup of boys is going to gain substantially more than the subgroup of girls.

Interpretation by Cox and McCullagh (1982)





Regression to the mean:

$$W_1 = a_0 + a_1 W_0 + \varepsilon$$
$$a_1 < 1$$

- Cox and McCullagh claim that both conclusions are correct
- Individual weights have changed, but the overall weight distribution is unaltered

- First statistician is correct for inferring the overall effect of diet
- Second statistician is correct in analyzing the individual changes in weight
  - Weight gain is negatively correlated with initial weight
  - Initially overweight individuals tend to lose weight, and conversely for initially underweight individuals → Cannot be inferred from a linear regression model
  - Paradox is resolved as it is inappropriate to compared males and females at fixed initial weight as we don't want to compare overweight females with underweight males

Interpretation by Holland and Rubin (1983)

Setup

- X_i: the initial weight in September
- $(Y_i(0), Y_i(1))$: potential weight in June is the student does not follow or follows the university's diet
- $G_i = g$: gender $g = 0, 1$

- Average effect of the university's diet: $\tau_g = E[Y_i(1) - Y_i(0) \mid G_i = g]$
- The effect difference between female and male: $\Delta = \tau_1 - \tau_0$

- Key issue leading to Lord paradox : $Y_i = Y_i(1)$ for all students, $Y_i(0)$ is never observed

Statistician 1:
If assuming $Y_i(0) = X_i$, then $Y_i(1) - Y_i(0) = Y_i - X_i$
$$\Delta = 0 - 0 = 0$$

Statistician 2:
From the data we know $Y(1) = Y = \alpha_g + bX + \varepsilon_1$
If we assume $Y(0) = a + bX + \varepsilon_0$,
then $Y(1) - Y(0) = \alpha_g - a + \varepsilon_1 - \varepsilon_0$
$$\Delta = \alpha_1 - \alpha_0 > 0$$

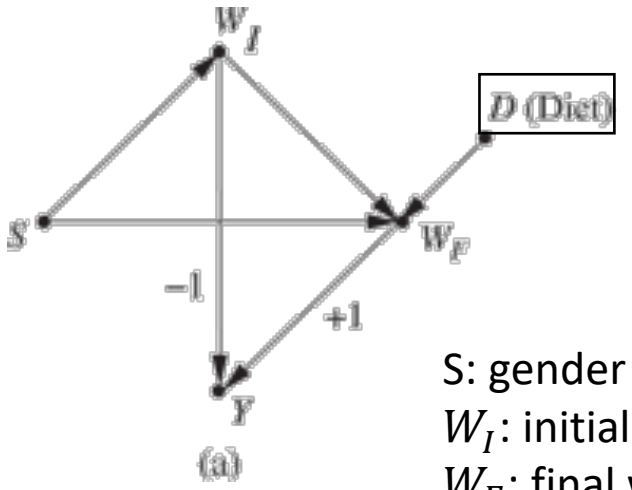
Interpretation by Pearl (2016)

- Lord original statement of interest: “The researcher wants to know how the groups would have compared if there had been no preexisting differences”
- Pearl thinks that “it is the effect of gender on weight gain that is the center of investigation while diet, since it is common to all subjects, should be treated as a fixed background condition.”

- Statistician 1: the perfect overlap of the two ellipses on the 45° line indicates that there is no difference in growth rate of the two sexes.
- Statistician 2: Adjust for the difference in initial weight, which may be attributed to the gender difference
- Dilemma: why should a greater weight gain (for men) which is found in every stratum of the initial weight WI suddenly disappear when averaged over the group as a whole.
- Similar to Simpson’s paradox: girls populate the underweight strata much more than boys

Interpretation by Pearl (2016)

- A causal mediation problem:
  - Initial weight mediates the causal process between gender and final weight
  - Statistician 1 estimates the total effect of gender on weight gain
  - Statistician 2 estimates the direct effect of gender on weight gain, adjust for the mediator



S: gender
W_I: initial weight
W_F: final weight
$Y = W_F - W_I$

Statistician 1:
$P(Y|do(S) = 1) - P(Y|do(S) = 0) = P(Y|S = 1) - P(Y|S = 0) = 0$

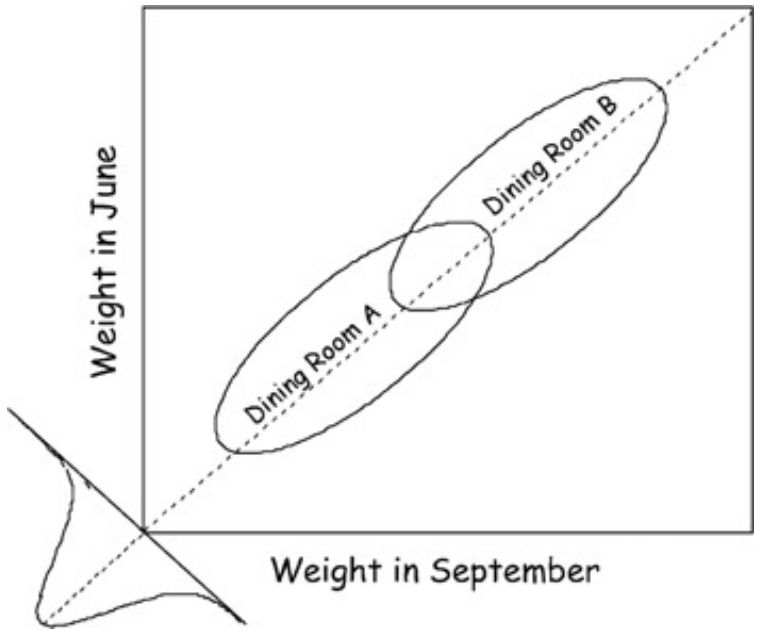
Statistician 2:
Assume structural equation

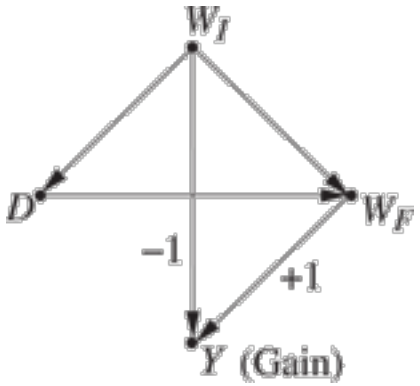
$$W_F = \alpha_0 + \alpha_1 W_I + \alpha_2 S + E_1$$

- Assume no exposure-outcome confounding
- Assume no mediator-outcome confounder
- Direct effect: CDE(m) = NDE = $\alpha_2 > 0$

Interpretation by Pearl (2016)

- A scenario where the graphical interpretation is more reasonable [Wainer and Brown, 2007]
  - Two different dining rooms serve two different diets
  - Students are randomly assigned into the residential hall last year
  - Each residential hall has a dining room that the students are required to go
  - The goal is to estimate the the causal effect of dining room on weight gain





- Initial weight is a confounder
- Statistician 1 is incorrect as he did not adjust for confounder
- Statistician 2 is correct if the structural equation
  $W_F = \alpha_0 + \alpha_1 W_I + \alpha_2 D + E_1$
  is correct